THE COVARIANCE MATRIX OF A CONTINUOUS AUTOREGRESSIVE VECTOR TIME-SERIES

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Notation. We shall denote by x^* or A^* the complex conjugate of the transpose of a column-vector x or of a square matrix A. Thus, $y^*x = \sum x_\nu \bar{y}_\nu$ is the scalar product of vectors x and y; but xy^* is a square matrix with components $x_\mu \bar{y}_\nu (\mu, \nu = 1, \cdots, n)$.

Theorem 1. *Let A be an $n \times n$ matrix of real or complex constants a_{rs}. Suppose that all of the eigenvalues of A lie in the left half-plane. Let $f(t)$ be an n-component column-vector satisfying*

$$Ef(t)f^*(t-\tau) = (Ef_r(t)\bar{f}_s(t-\tau)) = \delta(\tau)C$$

where $\delta(\tau)$ is the delta-function and where C is a positive semi-definite Hermitian matrix. Let $x(t)$ be the stationary stochastic process defined by

$$\frac{dx(t)}{dt} = Ax(t) + f(t) \qquad (-\infty < t < \infty). \tag{1}$$

Then

$$Ex(t)x^*(t-\tau) = e^{A\tau}M \tag{2}$$

where the $n \times n$ covariance matrix M is uniquely determined by the system of n^2 linear equations in n^2 unknowns

$$-C = AM + MA^*. \tag{3}$$

Proof. The steady-state solution of the differential equation (1) is

$$x(t) = \int_{-\infty}^{t} e^{(t-\lambda)A} f(\lambda)\, d\lambda = \int_0^\infty e^{\lambda A} f(t-\lambda)\, d\lambda.$$

Therefore,

$$\begin{aligned} Ex(t)x^*(t-\tau) &= E\int_0^\infty \int_0^\infty e^{\lambda A} f(t-\lambda) f^*(t-\tau-\mu) e^{\mu A^*}\, d\lambda\, d\mu \\ &= \int_0^\infty \int_0^\infty e^{\lambda A}\delta(\tau+\mu-\lambda) C e^{\mu A^*}\, d\lambda\, d\mu \\ &= \int_0^\infty e^{(\mu+\tau)A} C e^{\mu A^*}\, d\mu = e^{\tau A} M \end{aligned}$$

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