THE AUTOCORRELATION FUNCTION OF A SEQUENCE UNIFORMLY DISTRIBUTED MODULO 1

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1. Introduction. The study of sequences x_j uniformly distributed modulo 1 has been pursued mainly by number-theoreticians in the field of analytic number theory and diophantine analysis [2]. The properties of uniformly distributed sequences have recently become important in the application of Monte-Carlo methods and in the formation of random number generators and tables [3]. This paper proves a theorem which will facilitate the determination of the autocorrelation function of a uniformly distributed sequence.

The symbol $\{x\}$ denotes the *fractional part* of x, so that

$$0 \leqq \{x\} < 1. \tag{1.1}$$

The function $\{x\}$ is periodic with period 1 and assumes the value 0 only when x is integral. Let

$$\rho(x) = \tfrac{1}{2} - \{x\}, \tag{1.2}$$

then the function $\rho(x)$ is also periodic with period 1, assumes the value $\frac{1}{2}$ only when x is integral, and satisfies

$$-\tfrac{1}{2} < \rho(x) \leqq \tfrac{1}{2}. \tag{1.3}$$

Since $\rho(x)$ is a function of bounded variation, it possesses a convergent Fourier expansion which is

$$\rho(x) = \sum_{k=1}^{\infty} (\sin 2\pi kx/\pi k). \tag{1.4}$$

Let T denote the number of solutions of the inequalities

$$0 \leqq \{x_j\} \leqq \gamma, \qquad 1 \leqq j \leqq N, \qquad 0 \leqq \gamma < 1, \tag{1.5}$$

in which γ is fixed; then the sequence x_j is said to be uniformly distributed modulo 1 if and only if

$$T = \gamma N + o(N) \tag{1.6}$$

for each γ. An important criterion is that of Weyl [2] which states that the sequence x_j is uniformly distributed modulo 1 if and only if

$$\sum_{j=1}^{N} e^{i2\pi hx_j} = o(N), \qquad \text{for each integral } h \geqq 1. \tag{1.7}$$

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