POISSON COUNTS FOR RANDOM SEQUENCES OF EVENTS

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1. Introduction. We shall be concerned in this paper with the properties of random sequences of events, such as the arrivals of customers at a queue. If T_n denotes the instant of the nth event, then any such sequence, occurring in the time interval $(0, \infty)$, can be identified with the sequence of random variables

$$\mathfrak{I} = (T_1, T_2, \cdots) \tag{1}$$

satisfying

$$0 < T_1 \leqq T_2 \leqq \cdots \leqq T_n \leqq \cdots . \tag{2}$$

We make without further comment the assumption that only finitely many events occur in any finite time, so that

$$T_n \to \infty \qquad (n \to \infty). \tag{3}$$

There are a number of different ways of specifying the distributions of $\mathfrak{I}$, which are convenient in different contexts. In the theory of queues, for instance, it is usual to specify the distributions of the process $(t_1, t_2, \cdots)$, where

$$t_n = T_n - T_{n-1} . \tag{4}$$

(Here and elsewhere, we make the notational convention that $T_0 = 0$.) The process $\mathfrak{I}$ is called a *renewal sequence* if the t_n are independent and identically distributed; if in addition the t_n have a negative exponential distribution, $\mathfrak{I}$ is a *Poisson sequence*.

Another way of describing $\mathfrak{I}$ is in terms of the *counts* of the sequence in successive intervals. Thus we consider intervals $(0, a]$, $(a, 2a]$, $\cdots$, and denote by $C_n = C_n(a)$ the number of events in the nth interval:

$$C_n(a) = \text{number of } r \text{ with } (n-1)a < T_r \leqq na. \tag{5}$$

It is, however, clear that a knowledge of the distributions of the process $\{C_n(a)\}$ for any one value of a does not suffice to determine the distributions of $\mathfrak{I}$. Again, if $\mathfrak{I}$ is a renewal sequence, the structure of $\{C_n\}$ is, in general, exceedingly complex. For these and other reasons it seems that the count process $\{C_n\}$ is not well adapted to describe the sequence of events $\mathfrak{I}$.

Some of the disadvantages of the count process can be avoided by considering the counts of $\mathfrak{I}$ in intervals of unequal length, and this suggests considering the counts in intervals of random length. It will be shown in Section 7 that this apparently arbitrary procedure arises very naturally in the theory of queues, and it is suggested that the idea of a "randomized count process" may prove useful

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