INVARIANTS UNDER MIXING WHICH GENERALIZE DE FINETTI'S THEOREM: CONTINUOUS TIME PARAMETER[1]

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0. Introduction. In a previous paper [4], the problem of characterizing mixtures of certain families of discrete-time stochastic processes was solved. The analysis will now be extended to continuous time. At least under a suitable continuity condition, the present discussion gives necessary and sufficient conditions for a process to be a mixture of stationary Markoff chains, or of processes with stationary, independent increments. Roughly, the law of a process is a weighted average of laws of stationary Markoff chains if and only if the probability that the process passes through a given finite sequence of states at given times depends only on the initial state, the number of transitions between each pair of states, and the length of time these transitions take (Theorem 2). The law of a process is a weighted average of laws of processes with stationary, independent increments if and only if the process has exchangeable increments (Theorem 3). This theorem was obtained, for real-valued processes, by a different method in Section 4.3 of Bühlmann (1960). The law of a process is a weighted average of laws of Brownian motions if and only if the process is d-isotropic (Definition 6, Theorem 4). The law P of a process $\{X_t\}$ is a weighted average of laws of Poisson processes if and only if, for i_j non-negative integers and $-\infty < t_j < t_{j+1} < \infty$, $P(X_{t_j} \geqq X_{t_{j-1}}) = 1$ and $P[X_{t_j} - X_{t_{j-1}} = i_j, 2 \leqq j \leqq n] \prod_{j=2}^{n} i_j!(t_j - t_{j-1})^{-i_j}$ is a function of n, $t_n - t_1$ and $i_2 + \cdots + i_n$ alone (Theorem 5). Theorem 6 makes precise the idea that the law of a process is a weighted average of laws of Poisson processes if and only if the process has exchangeable increments and its sample functions are counting-functions (Definition 7), and Theorem 7 gives equivalent conditions in terms of holding times.

These results follow easily from Theorems 1 and D.1 of Section 2, which constitute a long and technical discussion of the Kriloff-Bogoliouboff (1937) theory (see also Oxtoby (1952)) in the appropriate probability space.

Here are two results for discrete-time processes. The first does not follow from the present study formally, but can be proved by obvious modifications of the argument. Let P be a probability on (the Borel subsets of) the space of bilateral sequences of real numbers. Then P can be represented as a weighted average of probabilities on this space under each of which the coordinates are independent and Gaussian with mean 0 and common variance if and only if under P any finite number of coordinates have a spherically symmetric joint distribution. The second result is proved as Lemma 10. The probability P can be represented as a weighted average of probabilities under each of which the coordinates are independent random variables with common exponential distribution if and

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