A WIENER-HOPF TYPE METHOD FOR A GENERAL RANDOM WALK WITH A TWO-SIDED BOUNDARY[1]

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1. Introduction. Let $\{z_n, n \geqq 0\}$ be a process of independent increments such that the increment $z_{n+h} - z_n$ has its distribution independent of n. Here, n and h run either through the nonnegative integers (discrete case) or through the nonnegative real numbers (continuous case). In the continuous case, the process $\{z_n\}$ is assumed to be separable, normalized in such a way that the sample functions are continuous to the right. Thus, in both cases the process is Markovian.

One has

$$E(e^{sz_n}) = E(e^{sz_0})E(e^{s(z_1-z_0)})^n = \hat{\sigma}(s)e^{n\theta(s)},$$

say. Let γ and v be real and *fixed* such that $\theta(\gamma) < v < \infty$. We shall be interested in finding explicit formulae for the generating functions

$$Q_1 = E(\{z_{N+0} \leqq 0\} \exp [sz_{N+0} - vN]), \qquad Q_2 = E(\{z_{N+0} \geqq c\} \exp [sz_{N+0} - vN]).$$

Here, $N = \inf \{n: z_n \not\in [0, c]\}$, while c denotes a fixed positive number. Further, s denotes a variable complex number with $\text{Re}\,(s) = \gamma$. In the discrete case, we shall usually write $e^{-v} = t$ and $e^{\theta(s)} = \varphi(s)$, thus, $\varphi(\gamma) < t^{-1}$.

As will be shown, such explicit formulae can be found in a large number of important special cases, where the *downward* jumps of the process $\{z_n\}$ are of such a simple type that the function $Q_1(s)$ is a priori known up to a finite number of parameters $a_1, \cdots, a_r$, (each depending on v or t but not on s).

In order to determine these parameters, we consider

$$Q_0(s) = \sum_{n=0}^{\infty} t^n E(e^{sz_n}\{N > n\})$$

in the discrete case, and $Q_0(s) = \int_0^\infty e^{-vn}E(e^{sz_n}\{N > n\})dn$ in the continuous case.

If B is a given Borel subset of the reals we denote by $M(B)$ the class of all complex-valued regular Borel measures μ supported by B such that the integral

$$\hat{\mu}(s) = \int_{-\infty}^{+\infty} e^{sy}\mu(dy), \qquad \text{Re}\,(s) = \gamma,$$

is absolutely convergent. The corresponding class of transforms $\hat{\mu}$ is denoted as $\hat{M}(B)$. Note that $Q_0 = \hat{\eta}$, where $\eta(D) = \sum_{n=0}^{\infty} t^n \Pr\,(z_n \,\varepsilon\, D, N > n)$ in the discrete case, similarly for the continuous case. Because $N > n$ implies $0 \leqq z_n \leqq c$, we have that the measure η is supported by the interval $[0, c]$, in other words $Q_0 \,\varepsilon\, \hat{M}([0, c])$.

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