CONDITIONED LIMIT THEOREMS[1]

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1. Introduction. Limit theorems for Markoff processes and suitable functionals defined on the processes occur in two principal contexts. The first category of applications treats a situation where the limit process is one of the classical stable processes. The usual approximating processes are sums of independent random variables. A second important class of examples is that of a limiting diffusion process of Bessel type (Section 2). Then the approximating processes may themselves either be of diffusion type (i.e., random walks, birth and death or bona fide diffusion) or processes almost of diffusion type [14], [23].

In both these cases under sufficient regularity conditions we have an invariance principle, i.e., the convergence of the processes entails the convergence in law of functionals continuous a.e. with respect to the limit process.

In this paper our objective is to develop several limit laws for random variables subject to conditioning on a recurrent event.

Such limit laws arise in a natural way in considering Kolmorogov-Smirnov statistics, and other related statistics as follows. Consider the Poisson process, $U(t)$, $t \geqq 0$, with stationary increments and $EU(1) = 1$. The event $U(n) = n$ for some n is a certain recurrent event ($n = 1, 2, \cdots$). Let N_n denote the number of recurrences that have taken place up to time n. It is well known that

$$\lim_{n\to\infty} \Pr(N_n/n^{\frac{1}{2}} < t) = (2/\pi)^{\frac{1}{2}} \int_0^t e^{-x^2/2}\, dx, \qquad 0 \leqq t < \infty, \text{[20], [3].}$$

On the other hand, $P(N_n = k \mid U(n) = n)$ is just the probability that $F_n(x) = F(x)$ for k values of x where F_n is the empirical c.d.f. of n independent random variables each distributed according to the c.d.f., F. It follows, in particular, from the results of this paper that

$$\lim_{n\to\infty} \Pr(N_n/n^{\frac{1}{2}} < t \mid U(n) = n) = 1 - e^{-t^2/2}, \qquad 0 \leqq t < \infty, \text{[20].}$$

Similarly, let $M_n = \max_{0 \leqq t \leqq n} (U(t) - t)$, $A_n = \max_{0 \leqq t \leqq n} |U(t) - t|$. The limiting distribution of $M_n/n^{\frac{1}{2}}$, $A_n/n^{\frac{1}{2}}$ are well known [11]. The "conditioned" versions involve the limits,

$$\lim_{n\to\infty} \Pr(M_n/n^{\frac{1}{2}} < t \mid U(n) = n), \quad \lim_{n\to\infty} \Pr(A_n/n^{\frac{1}{2}} < t \mid U(n) = n).$$

These, of course, are the well known limiting distributions of the one and two-sided Kolmogorov-Smirnov statistics, $n^{\frac{1}{2}}D_n^+$, $n^{\frac{1}{2}}D_n$. Similarly, instead of $U(t)$ one considers the process $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \cdots$, where the X_i are independent and identically distributed random variables equaling 1 and -1

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