GENERATING FUNCTIONS FOR MARKOV RENEWAL PROCESSES

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A general matrix representation is given for the multivariate transition probability generating functions of a Markov renewal process with a finite number of states. It is indicated how numerous derived probability distributions can be obtained by simple substitutions. Finally an application is made to the distribution of the maximum length of a busy $M/M/1$ queue.

Let $\mathbf{N}(t) = (N_1(t), \cdots, N_m(t))$ denote a Markov renewal process with a finite number m of states and with a matrix of transition probability distributions $\mathbf{Q} = \{Q_{ij}\}$ [3], [4]. Set $H_i = \sum_{j=1}^{m} Q_{ij}$. The $Q_{ij}(t)$ are nondecreasing right-continuous functions satisfying $Q_{ij}(0-) = 0$ and $H_i(\infty) = 1$ for all $i, j = 1, \cdots, m$. The random variable $N_i(t)$ is equal to the number of visits to state i during the time interval $[0, t)$. The stochastic process Z_t is referred to as the semi-Markov process (S-M P) associated with the Markov renewal process. $Z_t = i$ when state i is being visited at time t. We assume that $P\{Z_0 = i\} = p_i$ with $\sum_{i=1}^{m} p_i = 1$. Let $\mathbf{k} = (k_1, \cdots, k_m)$ denote an m-tuple of non-negative integers and define $T(\mathbf{k}) = \inf\{t: N_1(t) = k_1, \cdots, N_m(t) = k_m\}$, where it is interpreted to be $+\infty$ if the set is empty. Thus, $T(\mathbf{k})$ is the random time at which the Markov renewal process enters the state $\mathbf{k} = (k_1, \cdots, k_m)$. Let $Z(\mathbf{k}) = Z_{T(\mathbf{k})}$ and $T'(\mathbf{k})$ denote the time instant at which the S-M P leaves the state $Z(\mathbf{k})$. We define the following transition probabilities for the Markov renewal process.

$$C_j(\mathbf{k}, t) = P\{T(\mathbf{k}) \leqq t \text{ and } Z(\mathbf{k}) = j\} \tag{1}$$

and

$$D_j(\mathbf{k}, t) = P\{T(\mathbf{k}) \leqq t < T'(\mathbf{k}) \text{ and } Z_t = j\}. \tag{2}$$

It is clear that these distribution functions all vanish for $t < 0$. In the sequel we shall only consider them for nonnegative values of t. Let $\mathbf{e}_i$ denote the ith unit vector. The probabilities defined in (1) and (2) satisfy the following relations.

$$\begin{aligned} C_j(\mathbf{e}_i, t) &= \delta_{ij} p_j \\ C_j(\mathbf{k}, t) &= \sum_{\nu=1}^{m} C_\nu(\mathbf{k} - \mathbf{e}_j, t) * Q_{\nu j}(t) \text{ for } \mathbf{k} \neq \mathbf{e}_i \\ C_\nu(\mathbf{k} - \mathbf{e}_j, t) &\equiv 0 \text{ if } k_j = 0 \\ D_j(\mathbf{k}, t) &= [1 - H_j(t)] * C_j(\mathbf{k}, t). \end{aligned} \tag{3}$$

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