EXCHANGEABLE PROCESSES WHICH ARE FUNCTIONS OF STATIONARY MARKOV CHAINS[1]

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Let $\{V_n, -\infty < n < \infty\}$ be a stochastic process and let $W_n = \{V_k, -\infty < k \leqq n\}$. Then the W-process is a Markov process and the V-process is a function of the W-process. The W-process is stationary if, and only if, the V-process is stationary. Thus every stationary process is a function of a stationary Markov process. The W-process has a uncountable state-space even when the V-process has only two states. It is thus of some interest to isolate stationary processes which are functions of stationary Markov chains with a countable number of states. The present note solves this problem for exchangeable processes.

Let $\{Y_n, n \geqq 1\}$ be an exchangeable process (see [2], p. 365) with a countable state-space J. States of J will be denoted by δ and finite sequences of states of J will be denoted by s. For a sequence s of length n let $p(s) = P[(Y_1, \cdots, Y_n) = s]$. Let Q denote the space of all probability distributions on J. Each $q \varepsilon Q$ is then defined by a sequence $\{q(\delta), \delta \varepsilon J\}$ of non-negative real numbers which add up to 1.

From de Finetti's work [1] we know that $\{Y_n\}$ is a mixture of sequences of independent and identically distributed random variables with values in J. In our notation, this means that, if $s = \delta_1 \cdots \delta_n$, then

$$p(s) = \int_Q q(\delta_1) \cdots q(\delta_n) \, d\mu(q), \tag{1}$$

when μ is a probability measure on the Borel sets in Q. The measure μ is uniquely determined by the probability function p.

Theorem. *An exchangeable process $\{Y_n\}$ with a countable state-space J is a function of a stationary countable-state Markov chain if, and only if, it is a countable mixture of sequences of independent and identically distributed random variables with values in J.*

Proof. Let the measure μ be discrete that is, concentrated on a countable subset $\{q_\nu, \nu \geqq 1\}$ of Q. Let $a_\nu = \mu(\{q_\nu\})$. Then (1) can be written as

$$p(s) = \sum_\nu a_\nu q_\nu(\delta_1) \cdots q_\nu(\delta_n). \tag{2}$$

The state-space I of the underlying Markov chain is then defined as $I = \{(\nu, \delta) \mid \delta \varepsilon J, \nu \geqq 1\}$. Let M_ν be the square matrix with all its rows equal to q_ν and let M be the direct sum of the M_ν's. The probability measure $\mathbf{m}$ on I which gives a mass $a_\nu q_\nu(\delta)$ to the state (ν, δ) is a stationary initial distribution for M. Finally, let f be the function on I into J defined by $f[(\nu, \delta)] = \delta$.

Received 26 March 1963.

[1] Prepared with the partial support of the National Science Foundation, Grant GP-10.