NOTES

A LOCAL LIMIT THEOREM[1]

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0. Introduction. In this note we shall obtain asymptotic estimates for $c_n = P\{a \leqq S_n \leqq b\}$, where a and b are fixed, $a < b$, and $S_n = X_1 + \cdots + X_n$ is a sum of independent random variables with a common distribution and finite variance. It is shown that for X_1, nonlattice, with mean zero and variance σ^2, $c_n \sim (b - a)(2\pi n\sigma^2)^{-\frac{1}{2}}$. This will appear as an application of the central limit theorem of Cramér and Esseen. On the other hand, in the special lattice case (integer-valued X_1), we have $c_n = \nu[a, b](2\pi n\sigma^2)^{-\frac{1}{2}} + o(n^{-\frac{1}{2}})$, where $\nu[a, b]$ is the number of integers in $[a, b]$.

The theorems appear unified in the following formulation. If F_n is the measure induced on the real line by S_n, then $(2\pi n\sigma^2)^{\frac{1}{2}}F_n$ converges in the sense of distributions to Lebesgue measure λ in the nonlattice case, and to the measure ν in the special lattice case. Each of these measures may be viewed as a Fourier transform of Dirac's distribution. The remaining general lattice case will be treated similarly.

These results are simple but do not appear to have been published previously. They are related to some known results as follows: Under certain additional assumptions on distributions in the domain of attraction of a nonnormal stable law ($\sigma = \infty$), Gnedenko ([2], p. 236; [3]) has obtained strong local theorems. Our results complement those of Gnedenko in the lattice case ([2], p. 233).

1. Statement of results. If with probability one, X_1 has only the values $\alpha + k\beta$, $k = 0, \pm 1, \cdots$, with α and β fixed, then X_1 is said to have a lattice distribution. We suppose $EX_1 = 0$, $EX_1^2 = \sigma^2 < \infty$.

THEOREM 1. *If X_1 does not have a lattice distribution, then for all continuous functions, g, with compact support*

$$\int g(y)H_n\{dy\} \to \int g(y)\,dy, \tag{1.1}$$

where $H_n = (2\pi n\sigma^2)^{\frac{1}{2}}F_n$.

THEOREM 2. *If X_1 does not have a lattice distribution, then*

$$P\{a \leqq S_n \leqq b\} \sim (b - a)(2\pi n\sigma^2)^{-\frac{1}{2}}. \tag{1.2}$$

In the lattice case, assuming first that β is chosen maximally, we may set our

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