ON THE AXIOMS OF INFORMATION THEORY

By P. M. Lee

Churchill College, University of Cambridge

1. Introduction. The uniqueness of Shannon's measure of information is here proved under less restrictive conditions than previously.

Let $H_k(p_1, p_2, \cdots, p_k)(\sum p_j = 1$; all p's $> 0)$ be a measure of the information provided by the performance of an experiment with k possible outcomes of probabilities $p_1, p_2, \cdots p_k$ (c.f. Shannon [4] or Khinchin [3]).

We assume

(i) that H_k is permutation-symmetric for $k = 2, 3$; i.e. $H_2(t, 1 - t) = H_2(1 - t, t) = h(t)$, say, for $0 < t < 1$, and $H_3(p_1, p_2, p_3) = H_3(p_{\pi_1}, p_{\pi_2}, p_{\pi_3})$ for (π_1, π_2, π_3) any permutation of $(1, 2, 3)$ and any $p_1, p_2, p_3 > 0$ such that $p_1 + p_2 + p_3 = 1$;

(ii) that $h(\cdot)$ is a finite real-valued Lebesgue measurable function defined on $(0, 1)$ and that $h(\frac{1}{2}) = 1$ (previous authors have assumed $h(\cdot)$ continuous on $[0, 1]$, see Fadeev [1], monotone on $(0, \frac{1}{2})$ and on $(\frac{1}{2}, 1)$, see Kendall [2], or Lebesgue integrable on $[0, 1]$, see Tveberg [5]);

(iii) that for $0 < t < 1$, $k > 1$ and $p_1, p_2, \cdots p_k > 0$, $\sum p_j = 1$, we have $H_{k+1}(tp_1, (1 - t)p_1, p_2, \cdots, p_k) = H_k(p_1, \cdots, p_k) + p_1 H_2(t, 1 - t)$, so that $H_3(p_1, p_2\ \ p_3) = h(p_1 + p_2) + (p_1 + p_2)h(p_1/p_1 + p_2)$.

From (i) and (iii) we see that $h(\cdot)$ must satisfy the functional equations

(iv) $h(t) = h(1 - t)$,

(v) $h(p_1) + (1 - p_1)h(p_2/1 - p_1) = h(p_1 + p_2) + (p_1 + p_2)h(p_1/p_1 + p_2)$

(vi) $h(p_1) + (1 - p_1)h(p_2/1 - p_1) = h(p_2) + (1 - p_2)h(p_1/1 - p_2)$.

We shall show under assumptions (ii), (iv), and (v) that $h(t) = -t \lg t - (1 - t) \lg (1 - t)$ (lg denotes logarithm to base 2, log denoting logarithm to base e). It follows that H_k is uniquely determined for all k; in fact

$$H_k(p_1, p_2, \cdots, p_k) = -\sum p_i \lg p_i .$$

2. Simple lemmas. As in Zaanen [6] (Section 36, Theorem 1 and Lemma γ), we observe that if μ denotes Lebesgue measure in R_1, and if $\phi(\cdot)$ is a continuously differentiable increasing function with a strictly positive derivative which maps an open interval I onto an open interval $\phi(I)$, then ϕ maps Lebesgue subsets Q of I to Lebesgue subsets $\phi(Q)$ of $\phi(I)$, and

$$\mu(\phi(Q)) = \int_Q \phi'(t)\, dt.$$

From this we deduce:

Received 5 August 1963.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.*
www.jstor.org