LOWER BOUNDS FOR MINIMUM COVARIANCE MATRICES IN TIME SERIES REGRESSION PROBLEMS[1]

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1. Introduction. Linear regression problems in time series analysis have attracted considerable attention in the literature over the past decade and a half. Much of this work has been directed towards obtaining conditions under which least squares estimates have asymptotic efficiency 1 amongst the class of linear estimates, as for example, in [2], [3] and [5]. Asymptotic efficiency here refers to the limiting behavior of the covariance matrix of least squares estimates relative to the covariance matrix of minimum variance unbiased estimates as the observation set grows large, thus for the continuous parameter case, an observation set $[0, A]$ as $A \rightarrow \infty$. Over finite intervals, precise information about the relationship between these matrices is impossible to obtain unless one imposes a correlation structure of a very few special forms. Such questions beg answers as one does not usually know the best estimates or the minimum covariance matrix. Theorems of Parzen [8] give formal answers to these questions in terms of Gram matrices in appropriate reproducing kernel spaces. These theorems provide a uniform framework for such problems but do not usually give explicit solutions as they represent a restatement in terms of norms of functions. Since the framework is available, it is natural to inquire how the theory of these spaces may be used and we shall be concerned here with essentially one possibility of extending the norm structure for known examples to related classes of kernels. This will produce upper bounds on norms, which in turn will give lower bounds on minimum covariance matrices and lower bounds on efficiency. The classes of kernels for which these bounds are obtained are stationary and completely monotone or convex only. The latter case generalizes a result of Hájek [4] for the case of an unknown mean with a convex correlation function.

Section 2 is devoted to the basic inequality. Care is taken to state the result in some generality as we have not attempted to exhaust its potential here. In Section 3 we apply this to the above mentioned regression problems. Among the computed examples we show that the efficiency of the least squares estimate of the mean for an interval $[0, A]$ is at least $\frac{3}{4}$ if the correlation function is convex and is at least $\sim\frac{7}{8}$ if the correlation function is completely monotone, provided, in each case, the correlation function vanishes at infinity.

2. An inequality. Given a positive definite matrix R which is in the convex cone generated by the positive definite matrices K_α, and given a family of linearly independent vectors x_i, it is possible to bound above, in the sense of positive

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