ON SEQUENTIAL CONTROL PROCESSES[1]

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1. Introduction. Consider a dynamic system which at times $t = 0, 1, \cdots$ is observed to be in one of $L + 1$ states $0, \cdots, L$. After each observation the system is "controlled" by making one of K decisions $d_1, \cdots, d_K$. Let $\{Y_t\}$, $t = 0, 1, \cdots$, denote the sequence of observed states and $\{\Delta_t\}$, $t = 0, 1, \cdots$, the sequence of decisions. We shall assume that

$$P(Y_{t+1} = j \mid Y_0, \Delta_0, \cdots, Y_t = i, \Delta_t = d_k) = q_{ij}(k),$$
$$t = 0, 1, \cdots; \quad j = 0, \cdots, L; \quad k = 1, \cdots, K$$

where the $q_{ij}(k)$'s are non-negative numbers satisfying

$$\sum_{j=0}^{L} q_{ij}(k) = 1, \qquad i = 0, \cdots, L; \quad k = 1, \cdots, K.$$

A rule for making the successive decisions can be summarized in the form of a sequence of non-negative functions

$$D_k(Y_0, \Delta_0, \cdots, \Delta_{t-1}, Y_t), \qquad t = 0, 1, \cdots; \quad k = 1, \cdots, K,$$

where in every case $\sum_{k=1}^{K} D_k = 1$. We set

$$P(\Delta_t = d_k \mid Y_0, \Delta_0, \cdots, \Delta_{t-1}, Y_t) = D_k(Y_0, \Delta_0, \cdots, \Delta_{t-1}, Y_t)$$

for $t = 0, 1 \cdots; k = 1, \cdots, K$. Thus, given any rule R for making the successive decisions, the sequence $\{Y_t\}$, $t = 0, 1, \cdots$, is a stochastic process possessing a finite state space $0, \cdots, L$ with its probability measure dependent upon the way the rule brings into play the conditional probabilities $q_{ij}(k)$. In particular, if the rule R is of the form

$$D_k(Y_0, \Delta_0, \cdots, \Delta_{t-1}, Y_t = i) = D_{ik}, \qquad t = 0, 1, \cdots; \quad i = 0, \cdots, L,$$

where $\sum_{k=1}^{K} D_{ik} = 1$, $i = 0, \cdots, L$, then $\{Y_t\}$ is a finite state Markov chain with stationary transition probabilities

$$p_{ij} = \sum_{k=1}^{K} D_{ik} q_{ij}(k), \qquad i = 0, \cdots, L; \quad j = 0, \cdots, L. \tag{1.1}$$

Let C denote the class of *all* possible rules, C', the above class of randomized stationary type rules, and C'', the finite sub-class of C' for which the D_{ik}'s are either 0 or 1.

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