THE MOMENTS OF A VARIATE RELATED TO THE NON-CENTRAL t[1]

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1. Introduction. Suppose W is a random normal variate with mean θ and variance 1 and X^2 is independently distributed as chi-squared with n degrees of freedom. Then, the random variate Q with non-centrality θ and n degrees of freedom is defined by

$$Q = W/(W^2 + X^2)^{\frac{1}{2}}, \qquad \text{(all roots positive)}.$$

It is apparent that $n^{\frac{1}{2}}Q/(1 - Q^2)^{\frac{1}{2}}$ is distributed as non-central t with non-centrality θ and n degrees of freedom. Further, Q^2 is distributed as non-central beta, see for example Hodges (1955), with non-centrality θ and shape parameters $\frac{1}{2}$ and $\frac{1}{2}n$ (in Hodges' notation $\lambda = \frac{1}{2}\theta^2$, $a = \frac{1}{2}n$, $b = \frac{1}{2}$).

This paper gives closed form analytic expressions and recurrence relations for the moments about zero of Q. As well, Table 1 gives numerical values of the mean, variance, third and fourth central moments. The present interest in the variate Q derived from another study, Hogben et al. (1962b) in the course of which a knowledge of the moments of Q was desired. Other uses of the table are suggested in Section 6. The present authors know of no previous published work on the moments of Q. Studies of the probability integral of the general non-central beta have been published by Tang (1938), Thompson (1941), Lehmer (1944), Nicholson (1954) and Hodges (1955).

2. Analytic expressions for the moments of Q. By transforming the joint density of W and X to polar coordinates and integrating, the probability density function of Q, $n > 0$, is directly found to be

$$(1) \qquad \begin{aligned} f(q)\,dq &= K_{n,\theta}^{-1}(1 - q^2)^{\frac{1}{2}(n-2)}P_n(\theta q)\,dq, \qquad -1 \leqq q \leqq 1 \\ &= 0, \quad \text{elsewhere,} \end{aligned}$$

where

$$K_{n,\theta} = \pi^{\frac{1}{2}}2^{\frac{1}{2}(n-1)}\Gamma(\tfrac{1}{2}n)e^{\frac{1}{2}\theta^2}$$

and

$$P_n(z) = \int_0^\infty y^n e^{zy - \frac{1}{2}y^2}\,dy.$$

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