VÁCLAV E. BENEŠ, *General Stochastic Processes in the Theory of Queues*. Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1963. $5.75, £2.3.0. viii + 88 pp.

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This is a monograph dealing with the single-server queue, first from a very fundamental nonprobabilistic point of view, then, in more concrete fashion as specific hypotheses leading to concrete analytical results are introduced. The work will be of primary interest to the reader interested in the mathematical theory of the single-server queue, with strong emphasis on Beneš' own work, rather than the practitioner interested in a collection of formulas covering diverse physical situations. This is not to minimize, however, the practical value of the results in Chapters 4 and 5, to be mentioned below.

The emphasis is on obtaining as much information as possible about the quantity $W(t)$, the so-called *virtual* waiting time. $W(t)$ can be defined in a physical way as the length of time an "inspector" (not one of the customers) would have to wait if he arrived at the queue at time t, and joined the line of customers present at that time. The "load" on the queue is described by specifying the step function $K(\cdot)$ which jumps upward by an amount equal to the service period of a customer at the instant of his arrival. The general philosophy is to consider the queue as operating on the function $K(\cdot)$ to produce the function $W(\cdot)$. Mathematically, $W(\cdot)$ and $K(\cdot)$ are related through the integral equation

$$W(t) = K(t) - t + \int_0^t U[-W(u)]\,du, \qquad t \geqq 0, \tag{1}$$

where $U(x) = \max(0, \operatorname{sgn} x)$.

Historically, the very fruitful idea of focusing interest on $W(\cdot)$, rather than the waiting times of the individual customers, which are more difficult to handle, is due to Takács [5], who considered the case of non-uniform Poisson arrivals and independent service times. Later on ([3], [1], [2], [4]; the last of these references presumably appeared after completion of the manuscript) it was realized that interesting information regarding $W(\cdot)$ could be obtained from (1) without any probabilistic assumptions whatsoever regarding the nature of $K(\cdot)$. In this way, the foundations of the theory are laid in Chapters 1 and 2, and when, finally, probabilistic assumptions regarding $K(\cdot)$ are made, the significance of the functional equations for $\Pr\{W(t) \leqq w\}$ is brought out very clearly.

In order to make the functional equations tractable to analysis by transform methods, some kind of stationarity assumptions are necessary. Specifically, the assumption of weak stationarity, which states that

$$\Pr\{K(t) - K(u) - t + u \leqq w \mid W(u) = 0\}$$