# NOTES

## MEMORYLESS STRATEGIES IN FINITE-STAGE DYNAMIC PROGRAMMING[1]

By David Blackwell

*University of California, Berkeley*

Given three sets $X$, $Y$, $A$ and a bounded function $u$ on $Y \times A$, suppose that we are to observe a point $(x, y) \varepsilon X \times Y$ and then select any point $a$ we please from $A$, after which we receive an income $u(y, a)$. In trying to maximize our income, is there any point to letting our choice of $a$ depend on $x$ as well as on $y$? We shall give a formalization to this question in which sometimes there is a point. If $(x, y)$ is selected according to a known distribution $Q$, however, we show that dependence on $x$ is pointless, and apply the result to obtain memoryless strategies in finite-stage dynamic programming problems.

We suppose that $X$, $Y$, $A$ are Borel sets in Euclidean spaces and that $u$ is bounded and Borel measurable. A *strategy* $\sigma$ is a Borel measurable map of $X \times Y$ into $A$: $\sigma(x, y)$ is the $a$ selected by $\sigma$ when $(x, y)$ is observed. The *income from* $\sigma$ is the function $I_\sigma$ on $X \times Y$: $I_\sigma(x, y) = u(y, \sigma(x, y))$. A *memoryless strategy* $\tau$ is a Borel measurable function from $Y$ into $A$; its income is $I_\tau(x, y) = u(y, \tau(y))$. $I_\tau$ is defined on $X \times Y$, but depends on $y$ only.

*Question* 1. Given any $\sigma$, is there a $\tau$ with $I_\tau \geqq I_\sigma$ for all $(x, y)$?

If $A$ is finite, the answer is clearly yes: define $v(y) = \max_a u(y, a)$ and choose $\tau$ so that $u(y, \tau(y)) = v(y)$. Then, for any $\sigma$, $I_\sigma(x, y) \leqq v(y) = I_\tau(x, y)$.

If $A$ is countable, the answer is no, in an uninteresting $\epsilon$ sense. Here is an example: $X = \{1 - 1/n, n = 1, 2, \cdots\}$, $Y = \{0\}$, $A = X$, and $u(y, a) = a$. The $\sigma$ with $\sigma(x, y) = x$ has $I_\sigma(x, 0) = x$, so that $\sup_x I_\sigma(x, 0) = 1$. For any $\tau$, $I_\tau \equiv \tau(0) < 1$, so that there is an $x$ with $I_\sigma(x, 0) > I_\tau(x, 0)$. But for countable $A$, given any $\epsilon > 0$ (where $\epsilon$ can even be a Borel measurable function of $y$), there is a $\tau$ such that, for any $\sigma$, $I_\tau > I_\sigma - \epsilon$ for all $(x, y)$: put $v(y) = \sup_a u(y, a)$ and choose $\tau$ so that $u(y, \tau(y)) > v(y) - \epsilon$.

*Question* 2. Given any $\sigma$ and any $\epsilon > 0$, is there a $\tau$ with $I_\tau > I_\sigma - \epsilon$ for all $(x, y)$? Section 2.16 of [2] implies an affirmative answer with certain additional not very restrictive hypotheses. But here is an example where the answer is no. $X$ is a Borel subset of the unit square $R \times S$ whose projection $D$ on $R$ is not a Borel set. $Y = A =$ unit interval, and $u$ is the indicator of $X$:

$$u(y, a) = 1, \qquad \text{if } (y, a) \varepsilon X,$$
$$= 0, \qquad \text{if } (y, a) \not\varepsilon X.$$

For the strategy $\sigma$: $\sigma(x, y) = s$ for $x = (r, s)$, we have $I_\sigma((r, s), r) = u(r, s) = 1$, so that $I_\sigma$ is 1 on the subset $F$ of $X \times Y$ consisting of all points $((r, s), y)$ with $y = r$. But for any $\tau$, $I_\tau(x, y) = u(y, \tau(y))$. The projection of $G = \{(x, y): I_\tau(x, y) = 1\}$ on $Y$ is just the $y$-set $\{u(y, \tau(y)) = 1\}$, which is a Borel subset

Received 18 September 1963; revised 17 December 1963.

[1] Prepared with the partial support of the National Science Foundation, Grant GP-10.



Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.* www.jstor.org