SOME THEOREMS CONCERNING THE STRONG LAW OF LARGE NUMBERS FOR NON-HOMOGENEOUS MARKOV CHAINS

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1. Introduction.

1. *Preliminary.* This paper deals with the problem of finding (necessary or sufficient) conditions for the strong law of large numbers in the case of a Markov chain.

The results proved in this paper are of classical form, i.e. they come very close to those of Cantelli, Borel, Khintchine, Kolmogorov for mutually independent random variables; these classical results themselves remain true for a very large class of non-homogeneous Markov chains (for which $\alpha_i > \rho > 0$, $i \varepsilon I = (1, 2, \cdots)$). In the same way we obtain new results for homogeneous Markov chains ($\alpha_i = \rho > 0$, $i \varepsilon I$); these results contain as particular cases the analogous results for mutually independent random variables ($\alpha_i = 1$, $i \varepsilon I$).

A part of these results was announced in preliminary papers ([11]–[13]).

We express our results by means of *the ergodic coefficient of a stochastic transition function* ([2], [1]); in [9] can be found various of its definitions and properties that we shall use here.

2. *Notations and definitions.* Let $(\mathfrak{A}_i, \Sigma_i)$ be a measurable space, x_i the elements of $\mathfrak{A}_i$, A_i the measurable sets, elements in the σ-algebra $\Sigma_i (i \varepsilon I)$. If the sequence of random variables $\xi_i (i \varepsilon I)$ is a Markov chain, let us consider that it has the stochastic transition functions $P_i(x_i, A_{i+1})$ with domains of definition $(\mathfrak{A}_i, \Sigma_i, \mathfrak{A}_{i+1}, \Sigma_{i+1})(i \varepsilon I)$. We denote by $\alpha_i = \alpha(P_i)$ the ergodic coefficient of P_i and by $\alpha_{ij} = \alpha(P_{ij})$ that of the transition function $P_{ij}(x_i, A_j)$ for the time interval $i, j (i + 1 < j)$. We shall suppose that all the variances $D\xi_i (i \varepsilon I)$ are finite and we set

$$(1) \qquad \alpha^{(n)} = 1 - \eta_n = \min_{1 \leqq i < n} \alpha_i, \qquad D_n = \sum_{i=1}^{n} D\xi_i .$$

We assume that $\alpha_i > 0 (i \varepsilon I)$, because in many important formulae (Basic Lemma, Lemma 1, Theorem 1) $\alpha^{(n)}$ appears in the denominator.

The random variables $\sigma_i (i \varepsilon I)$ are called strongly stable, if there is some numerical sequence $d_i (i \varepsilon I)$ so that for $n \to \infty$, $\sigma_n - d_n$ converges to zero with probability 1. In this case it is possible, [7], to take $d_i = m\sigma_i$ (m—the median); the $\sigma_i (i \varepsilon I)$ are called *normally strongly stable* if it is possible to take $d_i = M\sigma_i$ (M—the expectation). Let

$$(2) \qquad S_n = \sum_{i=1}^{n} \xi_i , \qquad \sigma_n = n^{-1} S_n , \qquad \mathfrak{U}_n = \max_{1 \leqq s \leqq n} |S_s - MS_s|, \quad (n \varepsilon I).$$

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