SOME STRUCTURE THEOREMS FOR STATIONARY PROBABILITY MEASURES ON FINITE STATE SEQUENCES[1]

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1. Introduction. Given a stationary probability measure P on doubly infinite sequences over finitely many states, there are measures P_n induced on n consecutive coordinates. These measures P_n are again stationary to the extent that stationarity requirements are satisfied over subsets of n consecutive coordinates. Conversely, given a family of such "stationary" measures $\{P_n\}$ which are consistent, there is an induced stationary measure P on infinite sequences. Generally, given a fixed P_n, there are many stationary measures P which reduce to P_n. We shall consider the restrictions imposed on P by choosing P_n. In the main, our results relate the growth of the number of elements in the support of P_{n+k} to the choice of P_n.

In Section 2, we state some basic properties of these support numbers. Deterministic measures are defined and the structure imposed on P by a deterministic measure P_n is noted. In Section 3, beginning with a fixed P_n, we consider certain extremal measures P which reduce to P_n and show that there is always a deterministic measure P which reduces to P_n. In Section 4, we examine the largest possible growth of support numbers for a fixed P_n. An example is given of a measure P which has the property that its restrictions P_n are concentrated on precisely $n + 1$ vectors.

2. Preliminaries. Let X^n be the nth Cartesian product of the set $X = \{0, 1, \cdots, q - 1\}$, let X^I be the space of doubly infinite sequences $\{X(n)\}$ whose coordinates take values in the set X and let $\mathcal{A}$ denote the σ-field of subsets of X^I generated by the cylinder sets. Given a stationary probability measure P on $(X^I, \mathcal{A})$ we induce a probability measure on X^n by restricting P to sets of the form $\{x(1) = x_1, \cdots, x(n) = x_n\}$. This restricted measure will be denoted by P_n and usually abbreviated by writing

$$p_n(x_1, \cdots, x_n) = P\{x(1) = x_1, \cdots, x(n) = x_n\}.$$

For each choice of n, these measures satisfy

$$(*) \qquad \begin{aligned} p_n(x_1, \cdots, x_n) &= \sum_{x_0=0}^{q-1} p_{n+1}(x_0, x_1, \cdots, x_n) \\ &= \sum_{x_{n+1}=0}^{q-1} p_{n+1}(x_1, \cdots, x_n, x_{n+1}) \end{aligned}$$

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