ESCAPE PROBABILITY FOR A HALF LINE

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Let $\{X_n\}$, $n > 0$ be a sequence of independent, identically distributed random variables. Suppose $E|X_1| < \infty$ and $EX_1 > 0$. Then the strong law of large numbers assures us that from any initial point S_0, the random walk

$$S_n = S_0 + X_1 + \cdots + X_n$$

lies in the interval $(-\infty, 0]$ for at most a finite number of values of n with probability one, and thus the non-positive axis is a transient set for this Markov process. Let $M_n = \min(S_1, S_2, \cdots, S_n)$. As M_n is non-increasing, we have that $M = \lim_{n\to\infty} M_n = \inf_{n\geqq 1} M_n$, exists, and as $P(M = -\infty) \leqq P(S_n \leqq 0$ i.o.$) = 0$, we have that M is finite with probability one. Starting at a point x on the non-positive axis let us define the *escape function* $e(x)$ as the probability that a particle, initially at x, will in the first step enter the positive axis and thereafter never return to the non-positive axis. It will be convenient, and in accord with the potential theory for Markov processes, to define $e(x) = 0$ for $x > 0$. More precisely then, we define

$$\begin{aligned} e(x) &= P(M > 0 \mid S_0 = x) && \text{if } x \leqq 0 \\ &= 0 && \text{if } x > 0. \end{aligned}$$

Our principal aim in this note will be to establish the following result.

Theorem 1. *If* $E|X_1| < \infty$ *and* $EX_1 > 0$ *and if* Z *is the first positive partial sum starting from* $S_0 = 0$, *then*

$$e(x)/EX_1 = P(Z > -x)/EZ \qquad \text{if } x \leqq 0 \tag{1}$$

and

$$\int_{-\infty}^{\infty} e(x)\, dx = EX_1 . \tag{2}$$

Proof. For the sequence $\{S_n\}$ with $S_0 = 0$ let

$$\begin{aligned} W &= \inf\{k > 0 : S_k > 0\}, && \text{if for some positive integer } n,\ S_n > 0, \\ &= \infty, && \text{otherwise;} \\ W' &= \inf\{k > 0 : S_k \leqq 0\} && \text{if for some positive integer } n,\ S_n \leqq 0, \\ &= \infty, && \text{otherwise.} \end{aligned}$$

On the event $[W < \infty]$ let $Z = S_W$, and on the event $[W' < \infty]$ let $Z' = S_{W'}$. A basic identity in the fluctuation theory for sums S_n([4] Theorem III.6–

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