A THEOREM ON STOPPING TIMES

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Let (Ω, P^x, σ) be a time homogeneous strong Markov process with right continuous paths taking values in a locally compact space E with a countable base. The purpose of this note is to give a characterization of the Borel fields associated with stopping times for such a process.

For elaboration on the material in the next few paragraphs we refer the reader to [1] (where a slightly different sample space and notation are used). Let Δ be a point adjoined to E as the point at infinity if E is not compact and as an isolated point if E is compact. Let $\bar{E} = E \cup \Delta$ and let $\mathcal{B}$ and $\bar{\mathcal{B}}$ denote the topological Borel fields of E and $\bar{E}$ respectively. A real valued function f on E is always extended to $\bar{E}$ by the convention $f(\Delta) = 0$.

For the sample space Ω we take the set of all right continuous functions w from $[0, \infty)$ to $\bar{E}$ which also satisfy $w(t) = \Delta$ if $t \geqq \sigma(w)$, where $\sigma(w) = \inf\{t: w(t) = \Delta\}$. Given $t \geqq 0$ the mapping $w \to w(t)$ is denoted by $X(t)$ or $X(t, w)$, and $X: \Omega \to \Omega$ is the mapping $X(w)(t) = X(t, w)$. Let $\mathcal{F}^0(t)$ $(\mathcal{F}^0)$ denote the Borel field of subsets of Ω generated by the sets $X(s)^{-1}(B)$ with B in $\mathcal{B}$ and $s \leqq t$ $(s < \infty)$. If μ is a probability measure on $\mathcal{B}$ we define P^μ on $\mathcal{F}^0$ by $P^\mu(\Lambda) = \int P^x(\Lambda)\mu(dx)$ and define $\mathcal{F}$ to be the intersection over all such μ of the P^μ completions of $\mathcal{F}^0$. Define $\mathcal{F}(t)$ to be the Borel field consisting of those sets Λ such that for each probability measure μ on $\mathcal{B}$ there are sets A in $\mathcal{F}^0(t)$ and N in $\mathcal{F}$ such that $P^\mu(N) = 0$ and $(A - \Lambda) \cup (\Lambda - A) = N$. We note in passing that in some previous work we defined $\mathcal{F}(t)$ to be the intersection over all μ of the P^μ completions of $\mathcal{F}^0(t)$. In fact this gives an extension of $\mathcal{F}^0(t)$ that is a bit too restrictive; the extension given above is the one we should have used.

A function $T: \Omega \to [0, \infty]$ is called a stopping time if for every $t > 0$ the set $\{T < t\}$ is in $\mathcal{F}(t)$. The Borel field $\mathcal{F}(T)$ is then defined to be

$$\{\Lambda \in \mathcal{F}: \Lambda \cap \{T < t\} \in \mathcal{F}(t) \quad \text{for all } t\}.$$

The strong Markov property implies that $\mathcal{F}(t) = \bigcap_{t<s} \mathcal{F}(s)$ so that notation is consistent with what one gets by regarding a constant function as a stopping time.

Suppose now that T is a stopping time, and define the mapping $Y: \Omega \to \Omega$ by $Y(w)(t) = Y(t, w) = X(\min(t, T(w)), w)$. For each t in $[0, \infty)$ the mapping $w \to Y(t, w)$ is measurable relative to $\mathcal{B}$ and $\mathcal{F}(T)$ so that if $\mathcal{G}^0(T)$ denotes the Borel field generated by the sets $Y(t)^{-1}(B)$ with B in $\mathcal{B}$ and t in $[0, \infty)$ we have $\mathcal{G}^0(T) \subset \mathcal{F}(T)$. Consequently if $\mathcal{G}(T)$ consists of those sets which for each

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