THE TAIL σ-FIELD OF A MARKOV CHAIN AND A THEOREM OF OREY[1]

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1. Introduction. Orey (1962) proved that, if p is a transition probability matrix with one ergodic class of recurrent and aperiodic states, then

$$\lim_{n\to\infty} \sum_j |p^n(i_1, j) - p^n(i_2, j)| = 0.$$

We present here a somewhat different proof which may give additional insight. Of course, Orey's result implies the corollary to our Theorem 2, as well as our Theorem 1 and its corollaries.

Let $\{x_n : 0 \leqq n < \infty\}$ be a sequence of random variables on the probability triple $(\Omega, \mathfrak{F}, P)$. Let $\mathfrak{F}^{(n)}$ be the smallest σ-field over which the $x_\nu : \nu \geqq n$ are measurable. The *tail σ-field* $\mathfrak{F}^{(\infty)}$ of $\{x_n : 0 \leqq n < \infty\}$ is $\bigcap_{n=0}^{\infty} \mathfrak{F}^{(n)}$. The main result of this paper is: If $\{x_n : 0 \leqq n < \infty\}$ is a Markov chain with stationary transition probabilities, countable state space I, and all states recurrent, then $\mathfrak{F}^{(\infty)}$ is atomic under P. More precisely, let $\{I_c : c \varepsilon C\}$ be the partition of I into its cyclically moving subclasses. It is equivalent to the usual definition (Chung (1960) Section I.3) that i and j are in the same I_c if and only if there is an $n \geqq 0$ and a k in I with the n-step transition probabilities from i to k and from j to k both positive. Then each $\mathfrak{F}^{(\infty)}$-set differs from some union of sets $[x_0 \varepsilon I_c]$ by a set of P-measure 0. In particular, if $\{x_n : 0 \leqq n < \infty\}$ is aperiodic and has only one recurrent class, its tail σ-field is trivial and Orey's result follows. These results are proved in Section 2 which concludes by describing the tail σ-field of a random walk on a countable Abelian group, and the σ-field of exchangeable sets defined on the recurrent Markov chain $\{x_n : n \geqq 0\}$ with countable state space and stationary transitions. A set is *exchangeable* if it depends measurably on the $x_n : n \geqq 0$ and is invariant under finite permutations of them.

Section 3 contains three examples. Example 1 is an aperiodic chain with only one recurrent class, in which two independent particles, starting from different states, may never meet. Example 2 is a transient chain $\{x_n : 0 \leqq n < \infty\}$ with nonatomic tail σ-field but trivial *invariant σ-field*. An event A is *invariant* if there is a Borel set B of I-sequences, with $(i_0, i_1, \cdots)$ in B if and only if $(i_1, i_2, \cdots)$ is in B, and $A = (x_0, x_1, \cdots)^{-1}B$. Example 3 is a stationary, three state Markov chain with one aperiodic, ergodic class, but a non-trivial σ-field of exchangeable events.

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