SOME THEOREMS ON FUNCTIONALS OF MARKOV CHAINS

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1. Introduction. In this paper we shall investigate various phenomena associated with a Markov process in discrete time, extending results found in [3], [6], [7], and [13]. The paper is divided into three parts. In part one we focus our attention on recurrent events (i.e., to successive entrances into some fixed state of a Markov chain with the positive integers as states) and show that the waiting time distribution is completely determined by the sequence $\{EY_n\}$, where Y_n is the time as observed from n that the event last took place. Moreover, we show that criteria for the event to be persistent, transient, positive, etc., may be given directly in terms of the EY_n. In part two we examine a particular class of null events called β-regular (see Section 2 for the definition), where we find various joint limit distributions for some of the functionals usually associated with these events. In part three we extend these limit laws to situations more general than recurrent events, and these extended results are then applied to several concrete situations.

2. Criteria for recurrent events. Let e be a recurrent event on the positive integers with waiting times W_k, these being independent, identically distributed, positive integer-valued random variables which may also assume the value ∞. We recall that e is called transient or persistent according as to whether or not $\rho = P(W_1 < \infty) < 1$. A persistent event is positive if $EW_1 < \infty$ and is null if $EW_1 = \infty$. A recurrent event is periodic of period λ if e may only occur at times $n\lambda$ for $n = 0, 1, 2, \cdots$. If $\lambda = 1$ we say that e is aperiodic. From now on we shall always assume that the recurrent event is aperiodic. The methods needed to extend results to the case of periodic events are both simple and standard [8]. We introduce the following functions. For $n > 0$ let

$N_n = \sup\{k \leqq n: W_1 + \cdots + W_k \leqq n\}$ (number of occurrences by time n)

$Y_n = W_1 + \cdots + W_{N_n}$ (time of last occurrence),

$Y'_n = n - Y_n$ (time elapsed since last occurrence),

$Z_n = W_1 + \cdots + W_{N_n+1} - n$ (time to elapse until next occurrence). For $n = 0$ let $N_0 = Y_0 = 0$.

Let $u_n = P(Y_n = n)$, $q_n = P(Y_n = 0)$, and for $|t| < 1$ let

$$F(t) = E(t^{W_1}; W_1 < \infty) = \sum_{k=1}^{\infty} t^k P(W_1 = k),$$

$$U(t) = 1 + \sum_{n=1}^{\infty} u_n t^n,$$

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