POLYNOMIAL EXPANSIONS OF BIVARIATE DISTRIBUTIONS[1]

By G. K. Eagleson

University of Sydney

1. Introduction. Bivariate distributions subject to the condition of ϕ^2 boundedness, to be defined later, may be expanded in a canonical form (e.g. Lancaster [3]). In this paper, a class of bivariate distributions, whose canonical variables are the orthonormal polynomials of the marginal distributions, is exhibited. This class consists of the bivariate normal, a bivariate gamma, Poisson, negative binomial, binomial and hypergeometric.

The identity in Hermite-Chebyshev polynomials due to Runge [7] is generalised to this class of distributions, which have a particular type of generating function for the orthogonal polynomials. The completeness of the orthogonal polynomials of the same class is also proved.

The bivariate distributions are generated by considering sums of independent random variables, which are "additive" (i.e. closed under convolutions), where some of the variables are held in common. Pearson [6] generates the bivariate normal in this way and Cherian [1] a bivariate gamma.

The conditions on the parameters, in order that the distributions be ϕ^2 bounded, are obtained. The regressions are shown to be linear and the correlation coefficient is seen to be a satisfactory measure of dependence. Finally, a goodness of fit test for these bivariate distributions is outlined.

2. Distributions with a particular form of generating function for the orthogonal polynomials. Meixner [5] considered those distributions which have a generating function for their orthogonal polynomials of the form

$$G(t, x) = f(t)e^{xu(t)} = \sum_{n=0}^{\infty} P_n(x)t^n/n! \tag{2.1}$$

where $P_n(x) = x^n + a_{n,1}x^{n-1} + \cdots + a_{n,n}$, $f(t)$ is a power series in t with $f(0) = 1$ and $u(t)$ is a power series in t with $u(0) = 0$ and $u'(0) = 1$. Both $f(t)$ and $u(t)$ have real coefficients. We denote the functional inverse of $u(t)$ by $v(u)$, i.e. $v(u(t)) \equiv t$. When needed, subscripts will be used on $f(t)$ and $u(t)$ to indicate with which distribution they are associated. Thus

$$\begin{aligned} &\textstyle\int_{-\infty}^{\infty} f(t)f(s) \exp\{xu(t) + xu(s)\}\, d\psi(x) \\ &\qquad = f(t)f(s)M(u(t) + u(s)) = \sum_{i=0}^{\infty} [(ts)^i/(i!)^2]c_i \end{aligned} \tag{2.2}$$

where $c_i = \int_{-\infty}^{\infty} P_i^2(x)\, d\psi(x)$, $\psi(x)$ is the distribution function of X, and $M(\theta) = E(e^{X\theta})$.

Received 27 November 1963.

[1] This research was undertaken while holding a C.S.I.R.O. Studentship at the University of Sydney.