EIGENVALUES OF NON-NEGATIVE MATRICES[1]

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1. Introduction. Let $P = (p_{ij})$, $i, j = 0, 1, 2, \cdots$, be a matrix with non-negative entries. P is said to be irreducible if for every pair i, j, there is a finite sequence of integers $k_1, k_2, \cdots, k_n$ such that $p_{ik_1}p_{k_1k_2} \cdots p_{k_nj} > 0$. An alternative definition is given in Gantmacher's book ([4], p. 50).

The point of view adopted here is to consider an irreducible matrix P as an operator acting on column vectors having non-negative entries. A necessary and sufficient condition for there to be a solution of $Px = \lambda x$, i.e. for λ to be an eigenvalue, is obtained. The principal tool is the theorem of Harris [5] and Veech [7] which gives a necessary and sufficient condition for the existence of a stationary measure for a transient Markov chain.

The relationship between the R-recurrent matrices studied by Vere-Jones [8] and Kingman [6] and recurrent matrices is investigated in the final section. In the stochastic case, this investigation is related to the eigenvalue problem described above.

2. The eigenvalue problem. The method to be used is to transform P into a substochastic matrix so that the Harris-Veech theorem may be applied. The first step is the observation that an eigenvector can only have positive components.

Lemma 1. *If P is irreducible and $sPx \leqq x$ for some $s > 0$ and nontrivial x, then $x_j > 0$ for all j.*

Proof. For any i, j, $sp_{ij}x_j \leqq x_i$, so by induction for any sequence $\{k_n\}$, $s^{n+1}p_{k_0k_1}p_{k_1k_2} \cdots p_{k_nk_{n+1}}x_{k_{n+1}} \leqq x_{k_0}$. Now $x_j > 0$ for some j and then for any i let $\{k_n\}$ be the sequence guaranteed by the definition of irreducibility. Letting $k_0 = i$, $k_{n+1} = j$ in the above inequality yields the positivity of x_i.

The next lemma proves the existence of all iterates of P provided there is an eigenvalue. Let $p_{ij}^{(0)} = \delta_{ij}$, $p_{ij}^{(n)} = \sum_k p_{ik}p_{kj}^{(n-1)}$, $P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)}s^n$, and R_{ij} equal the radius of convergence of this power series.

Lemma 2. *If P is irreducible and $sPx \leqq x$ for some $s > 0$ and nontrivial x, then $p_{ij}^{(n)} < \infty$ for all i, j, and n, and $R_{ij} \geqq s$.*

Proof. First $sp_{ij}x_j \leqq x_i$ and using the given inequality in an induction yields $s^n p_{ij}^{(n)}x_j \leqq x_i$. This suffices for the first part since $x_j > 0$ by Lemma 1. Finally $\{p_{ij}^{(n)}\}^{1/n} \leqq s^{-1}(x_i/x_j)^{1/n}$ so that $\limsup \{p_{ij}^{(n)}\}^{1/n} \leqq s^{-1}$ or $R_{ij} \geqq s$.

In view of Lemma 2 and our interest in P having eigenvalues, it will be assumed henceforth that all iterates of P are finite and that $R_{ij} > 0$. Vere-Jones [8] has shown in this case that $R_{ij} = R$, independent of i and j, and that the series $P_{ij}(R)$ converge or diverge together. In the first case P is called R-transient

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