ON ASYMPTOTIC MOMENTS OF EXTREME STATISTICS

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1. Introduction. Let Y_n be the maximum (or minimum) of n independent observations of a random variable X. On pp. 87–89 of his book, Gumbel [1] stated that it was still not known for general cases how the mean and standard deviation of Y_n depend on the sample size n and that such knowledge would have important applications. A later paper by Sen [2] contains some results of this type, but only for cases in which the distribution of X has a finite upper (or lower) end point.

A method for establishing the asymptotic behavior (as $n \to \infty$) of moments of Y_n is illustrated in this paper. It also implicitly provides formulas that can be used to compute lower and upper bounds on these moments, for finite values of n. The method involves the introduction of an auxiliary variable y and a double limit, first on n and then on y.

Of the three general cases treated here, only one requires that the distribution of X have a finite upper end point. Although none of them permit X to have a normal or general gamma distribution, the author believes and hopes to show later that some of the results can be extended to include these important cases. The results are presented in the next section and are proved in the last section.

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2. The results. Let F be the right-continuous probability distribution function of X, and let a_2 be the value, either finite or $+\infty$, for which $F(a_2) = 1$ and $F(x) < 1$ for $x < a_2$. For simplicity, we assume throughout that F has a continuous derivative f on some open interval (a_1, a_2). No detailed knowledge of $F(x)$ for $x < a_1 < a_2$ is required, since the asymptotic behavior of a moment of Y_n (if it exists) depends only on the properties of F in a neighborhood of a_2.

Hereafter, μ_n and σ_n^2 denote the mean and variance of Y_n, and $\lambda_k = E(|X - a|^k)$, where a is a constant to be specified. As usual, E and Γ denote the expectation operator and the gamma function. Of course, $x \to \alpha-$ indicates that x approaches α always with $x < \alpha$. Also, $g(x) \sim h(x)$ as $x \to \alpha-$, with finite α or $\alpha- = +\infty$, signifies that $g(x)/h(x)$ possesses a limit as $x \to \alpha-$ and that this limit is 1. Similarly, $A_n \sim B_n$ signifies that A_n/B_n possesses the limit 1 as $n \to \infty$.

Theorem 1. *If there are real constants a, $b > 0$, and $c > 0$ such that $F(a) = 1$, $F(x) < 1$ for $x < a$, and, as $x \to a-$,*

$$1 - F(x) \sim b(a - x)^c, \tag{2.1}$$

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