ON EXTREME ORDER STATISTICS

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1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with the common distribution function $F(x) = \Pr(X_i \leqq x)$. Define the order statistics Y_n^k for $n \geqq k$ by

$$Y_n^k = k\text{th largest among } (X_1, X_2, \cdots, X_n). \tag{1.1}$$

(We will usually write M_n for Y_n^1, the maximum.) The random variables Y_n^k have, of course, been the subject of many papers, and in particular their limiting behavior as $n \to \infty$ has been thoroughly investigated. A survey of results in this area (with some new ones) was recently published by Barndorff-Nielsen [1]. However, it does not seem that $\{Y_n^k\}$ has previously been studied *as a stochastic process* (with n for the parameter), and to make such a study, with emphasis on limit theorems, is the object of the present paper.

The problem of limiting distributions for M_n has been treated very completely by Gnedenko in [3]. He determined all the non-degenerate distribution functions $G(x)$ which can appear in

$$\lim_{n\to\infty} \Pr[(M_n - a_n)/b_n \leqq x] = G(x) \tag{1.2}$$

for some choice of the constants $b_n > 0$, a_n; such a function must be of the same type as one of the laws Φ_α, Ψ_α or Λ defined in [3]. (These distributions can be conveniently found in [2], Equation (1.1).) Conditions on F insuring that (1.2) holds are also given in [3]. In the present work, the form of the law $G(x)$ will usually not be important and it is convenient to simply take (1.2) as the basic hypothesis. (But it is useful to note that all limit distribution functions are continuous.)

Following a procedure which is common enough in other contexts we define the stochastic processes

$$m_n(t) = (M_{[nt]} - a_n)/b_n, \qquad t \geqq 1/n, \tag{1.3}$$

where [u] means the greatest integer not exceeding u; it is technically convenient to define $m_n(t) = m_n(1/n)$ for $0 \leqq t \leqq 1/n$. We shall show in the next section that whenever (1.2) holds,

$$\lim_{n\to\infty} \{m_n(t)\} = \{m(t)\} \tag{1.4}$$

in the sense of convergence of finite-dimensional (f.d.) joint distributions; $\{m(t)\}$ is a Markov process with increasing path functions. In a sense, there is only one process $\{m(t)\}$ despite the variety of possible limit laws G. In Section 3 the result (1.4) is strengthened by proving two versions of an "invariance principle",

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