ON THE COMPLEX WISHART DISTRIBUTION

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1. Introduction. Goodman [2] derived the complex Wishart distribution with the aid of characteristic functions and Fourier transforms. In the present paper we give a direct and simplified method of deriving this distribution. At the same time Lemma 13.3.1 of Anderson [1] is generalized to matrices with complex elements. This generalization leads to a straightforward extension of the results of Chapter 13 of Anderson [1] to complex matrices.

2. Preliminaries and definitions. For a complex number $z = x + iy$, $\bar{z}$ denotes the conjugate. A matrix M of elements m_{jk} is denoted by $\|m_{jk}\|$, the determinant of a square matrix by $|M|$, the transpose by M'.

For notational convenience, we shall not distinguish between a random variable and its observed values.

The pdf of a p-variate complex Gaussian distribution $\xi' = (z_1, z_2, \cdots, z_p)$ is

$$p(\xi) = \pi^{-p}|\Sigma_\xi|^{-1} \exp(-\bar{\xi}'\Sigma_\xi^{-1}\xi),$$

assuming each of the random variables x and y to have zero mean (See Goodman [2] for definitions).

Lemma 1. *If Y is a matrix of complex elements, of order $p \times m$, $p \leqq m$, and of rank p, then there exists a unique triangular matrix T with real and positive (>0) diagonal elements and a semi-unitary matrix L, $L\bar{L}' = I$, such that*

$$Y(p \times m) = T(p \times p)L(p \times m).$$

Proof. The proof is quite simple and is omitted.

3. Main results. The following theorem is a generalization of Lemma 13.3.1 of Anderson ([1], p. 319) to a matrix Y with complex elements.

Theorem 1. *If the density of $Y(p \times m)$ is $f(Y\bar{Y}')$, then the density of $B = Y\bar{Y}'$ is*

$$\{|B|^{m-p}f(B)\pi^{p[m-\frac{1}{2}(p-1)]}\}/[\textstyle\prod_{i=1}^{p}\Gamma(m - i + 1)]. \tag{1}$$

Proof. From Lemma 1, we can write

$$Y = TL, \tag{2}$$

where T is a triangular matrix with positive (>0) diagonal elements, and L a semi-unitary matrix.

We shall now find the Jacobian of the transformation, $J(Y \to T, L)$. Differentiating both sides of Equation (2), we have $(dY) = (dT)L + T(dL)$. Premultiplying by T^{-1}, we get $T^{-1}(dY) = T^{-1}(dT)L + (dL)$. Putting $U = T^{-1}(dY)$, and $V = T^{-1}(dT)$, we have $U = VL + dL$.

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