ROBUSTNESS OF THE HODGES-LEHMANN ESTIMATES FOR SHIFT[1]

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0. Introduction. Let $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ be $(m + n) = N$ independent observations from continuous distributions

$$P(X_i \leqq u) = F(u) \quad \text{and} \quad P(Y_j \leqq u) = F(u - \Delta), \qquad (i = 1, \cdots, m; j = 1, \cdots, n). \tag{0.1}$$

Two problems often studied in this setup are (i) either to test the hypothesis $\Delta = 0$ against $\Delta > 0 (\Delta \neq 0)$ or (ii) to estimate the shift Δ.

The classical approach to the testing problem is based on the statistic $t = (\bar{Y} - \bar{X})/\{(1/m + 1/n)[\sum_i (X_i - \bar{X})^2 + \sum_j (Y_j - \bar{Y})^2]/(m + n - 2)\}^{\frac{1}{2}}$ known to be approximately normally distributed under general assumptions on F. Similarly the classical estimate for Δ is

$$\hat{\Delta} = \bar{Y} - \bar{X}; \qquad \bar{Y} = \textstyle\sum_j Y_j/n, \qquad \bar{X} = \textstyle\sum_i X_i/m. \tag{0.2}$$

Both these methods are known to be vulnerable to gross errors. For the testing problem, rank tests, such as the Wilcoxon and the Normal Scores (Fisher-Yates) tests have been in use for several years and shown to be more robust against gross errors than the classical one. At the same time little efficiency (in the Pitman sense) is lost when after all no gross errors are present and normality assumptions hold.

Similar robust methods for the corresponding estimation problem have recently been proposed by J. L. Hodges, Jr. and E. L. Lehmann [5]. They study small-sample as well as large-sample properties of a large class of such estimates, derived from corresponding test statistics $h(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$ for the hypothesis $\Delta = 0$ against $\Delta > 0$, satisfying the following two conditions:

(A) $h(x_1, \cdots, x_m, y_1 + a, \cdots, y_n + a)$ is a nondecreasing function of a for all x and y, and such that

(B) when $\Delta = 0$, the distribution of $h(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$ is symmetric about a fixed point μ (independent of F),

(i) for all $F \in \mathfrak{F}_0$, or (ii) for all $F \in \mathfrak{F}_1$.

(Throughout this paper we shall use the same short notation that was used in [5]. Furthermore $P_{\Delta_0}(\cdot)$, $E_{\Delta_0}(\cdot)$, etc. will be used to indicate that the expression in question is being computed for the case $\Delta = \Delta_0$.)

Here as throughout this paper, $\mathfrak{F}_0$ denotes the class of all continuous distributions and $\mathfrak{F}_1$ the class of all continuous distributions symmetric about zero.

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