A RECURRENCE FOR PERMUTATIONS WITHOUT RISING OR FALLING SUCCESSIONS

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1. Introduction. For n elements, the rising successions in question are $12, 23, \cdots, \overline{n-1n}$; the falling successions are $21, 32, \cdots, \overline{nn-1}$. The enumeration of the permutations of the title has been considered by Irving Kaplansky [1] in the form of what he calls the "n-kings problem": in how many ways may n kings be placed on an n by n chessboard so that no two attack each other? In a later paper [2], he has treated the more general problem of enumerating permutations of n elements by the number of successions of either kind (more briefly, by the number of instances in which i is next to $i+1$, $i = 1, 2, \cdots, n-1$). If S_{nk} is the typical number of such an enumeration, $S_n(t) = \sum S_{nk}t^k$ is called the enumerator (of permutations by number of successions); $S_n(t)$ is a polynomial in t of degree $n-1$.

It will be shown that

$$(1) \quad S_n(t) = (n+1-t)S_{n-1}(t) - (1-t)(n-2+3t)S_{n-2}(t) - (1-t)^2(n-5+t)S_{n-3}(t) + (1-t)^3(n-3)S_{n-4}(t), \quad n > 3$$

with $S_0(t) = S_1(t) = 1$, $S_2(t) = 2t$, $S_3(t) = 4t + 2t^2$. Recurrence (1) has the particular virtue of reducing to the following pure recurrence for the numbers of the title, $S_n = S_n(0)$:

$$(2) \quad S_n = (n+1)S_{n-1} - (n-2)S_{n-2} - (n-5)S_{n-3} + (n-3)S_{n-4}, \quad n > 3.$$

2. Preliminary résumé. The results of [1] and [2] needed for present purposes are as follows:

$$(3) \qquad S_n(t) = \textstyle\sum_{k=0}^n A_{nk}(n-k)!(t-1)^k,$$

where

$$(4) \qquad A_{nk} = A_{n-1,k} + A_{n-1,k-1} + A_{n-2,k-1}, \qquad n > 1$$

or

$$(5) \qquad A_n(x) = \textstyle\sum_{k=0}^n A_{nk}x^k = (1+x)A_{n-1}(x) + xA_{n-2}(x)$$

where, by convention, $A_0(x) = A_1(x) = 1$. It following at once from (3) and (4) that (primes denote derivatives)

$$(6) \quad S_n(t) = (n-1+t)S_{n-1}(t) + (1-t)S'_{n-1}(t) - (n-1)(1-t)S_{n-2}(t) - (1-t)^2S'_{n-2}(t), \quad n > 1$$

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