LOCAL CONVERGENCE OF MARTINGALES AND THE LAW OF LARGE NUMBERS

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0. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and $\mathfrak{F}_n$ be increasing Borel subfields of $\mathfrak{F}$. $(y_n, \mathfrak{F}_n, n \geqq 1)$ is said to be a stochastic sequence if y_n is extended real valued and $\mathfrak{F}_n$-measurable for each n. A stochastic sequence $(y_n, \mathfrak{F}_n, n \geqq 1)$ is called a submartingale (or martingale), if $E(y_n)$ exists (that is $Ey^+ < \infty$ or $Ey^- < \infty$) and $E(y_{n+1} \mid \mathfrak{F}_n) \geqq y_n$ (or $E(y_{n+1} \mid \mathfrak{F}_n) = y_n$) a.e. for each n. A stopping variable t is an extended positive integer valued random variable such that the set $[t = n] \varepsilon \mathfrak{F}_n$ for each positive integer n. For an extended real number a, define $a^+ = \max(0, a)$ and $a^- = \max(0, -a)$. For a set A, $I(A)$ denotes the characteristic function of the set A.

Recently, Neveu ([8], p. 143) proves a new submartingale convergence theorem, namely if $(s_n, \mathfrak{F}_n, n \geqq 1)$ is a submartingale with $E(s_n^+) < \infty$, then s_n has a limit a.e. where $\sum_2^\infty (E(s_n^+ \mid \mathfrak{F}_{n-1}) - s_{n-1}^+) < \infty$. Neveu's result suggests the present paper. In Section 1 we will generalize his result and in Section 2 prove a local convergence theorem of martingales, which extends a result of Loève [7] and improves a result of Lévy-Doob ([4] p. 320). Section 3 is denoted to the law of large number and a result due to Lévy-Neveu ([5]; [8], p. 141) is extended in this section.

1. Local convergence of submartingales.

Theorem 1. *Let $(s_n, \mathfrak{F}_n, n \geqq 1)$ be a submartingale with $E(s_1^-) < \infty$, and $(z_n, \mathfrak{F}_{n-1}, n \geqq 2)$ and $(y_n, \mathfrak{F}_n, n \geqq 2)$ be two stochastic sequences such that y_n is finite valued for each n. Let $z_1 = y_1 = 0$ and*

$$(1) \qquad s_n \leqq z_n + y_n, \qquad n \geqq 2.$$

For $b > 0$, let t be the first n such that $y_n > b$ and let

$$(2) \qquad E(y_t I[t < \infty]) < \infty.$$

Then s_n converges a.e. *where*

$$(3) \qquad \sup z_n < \infty, \qquad \sup y_n < b.$$

Proof. For any fixed $a > 0$, let t' be the first n such that $z_n > a$ and $w_n = \min(t' - 1, t, n)$. Then w_n is a sequence of bounded stopping variables and $w_n \leqq w_{n+1}$. Hence ([4], *p.* 303) s_{w_n} is a submartingale, since $E(s_1^-) < \infty$ implies that $E(s_n^-) < \infty$ for each n. Now

$$E(s_{w_n}^+) \leqq E(z_{w_n}^+) + \textstyle\sum_{j=1}^n E(y_j^+ I[w_n = j])$$

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