THE BEHAVIOR OF LIKELIHOOD RATIOS OF STOCHASTIC PROCESSES RELATED BY GROUPS OF TRANSFORMATIONS

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Introduction. Let x_α be a 1-parameter family of stochastic processes and P_α the associated probability measures on the space of sample functions. We assume that the x_α are gotten from x_0 by the application of a group T_α of transformations, i.e., that T_α is a group of automorphisms on an algebra F, of bounded measurable functions dense in $L_1(P_0)$ and that $\int T_\alpha f\, dP_0 = \int f\, dP_\alpha$ for all f in F and all α.

In Section 2 we classify these problems as being conservative, dissipative, or mixed in analogy with terminology of ergodic theory. It turns out that many problems of interest are dissipative. Section 3 contains several such examples. Section 4 gives results on the spectrum of the associated isometries of $L_s(P_0)$ and on the asymptotic behavior of $dP_\alpha(x)/dP_0$ in the dissipative case.

2. The conservative and dissipative sets. Throughout this paper we will assume that the P_α are mutually absolutely continuous, that the T_α preserve bounds and either

(1) $T_\alpha f(x)$ has a continuous derivative $D(T_\alpha f)(x)$ in α which is bounded uniformly in α and x for every f in F and every x, or

(2) $T_\alpha f$ has an L_1-continuous L_1-derivative $DT_\alpha f$ for every f in F and $\|DT_\alpha f\| = O(e^{K|\alpha|})$ for some K independent of f.

We shall write P for P_0.

It has been shown [for condition (1) see [4] (Theorem 1, p. 272) and for condition (2) see [5] (Theorem 3.3)] that the above conditions imply that T_α can be extended to a group of automorphisms of all measurable functions and that the maps of $L_1(P)$, defined by

$$V_\alpha f = (dP_\alpha/dP),\ T_{-\alpha} f$$

form a strongly continuous 1-parameter group of isometries.

Thus $dP_\alpha/dP = V_\alpha(1)$ is L_1 continuous and it follows that we may regard dP_α/dP as a measurable stochastic process. By Fubini's theorem then $\int_{-T}^{T} [dP_\alpha(x)/dP]\, d\alpha$ exists and is finite for every finite T for almost all x. Set

$$q(x) = \int_{-\infty}^{\infty} [dP_\alpha(x)/dP]\, d\alpha.$$

We define the *conservative* set C to consist of those x with $q(x) = \infty$ and the *dissipative* set D to consist of those x with $q(x) < \infty$.

Lemma 2.1. *The sets C and D are invariant under the T_α, to within sets of meas-*

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