ON THE ASYMPTOTIC THEORY OF FIXED-SIZE SEQUENTIAL CONFIDENCE BOUNDS FOR LINEAR REGRESSION PARAMETERS[1]

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1. Introduction. Chow and Robbins [3] have considered the problem of finding a confidence interval of prescribed width $2d$ and prescribed coverage probability α for the unknown mean μ of a population Ω having fixed distribution function F with unknown, but finite, variance $\sigma^2 > 0$. Since no fixed sample procedure can possibly work, they consider a certain class of sequential procedures and show that the members of this class are asymptotically "consistent" (i.e., cover μ with probability α) and asymptotically "efficient" (i.e., have expected sample size equal to the smallest sample size one could use if σ^2 were known) as d goes to zero. The purpose of this paper is to extend these results to the linear regression problem.

2. The problem. Consider $y_1, y_2, \cdots$ a sequence of independent observations with

$$y_i = \beta x^{(i)} + \epsilon_i, \tag{2.1}$$

β an unknown $1 \times p$ vector, $x^{(i)}$ a known $p \times 1$ column vector, and ϵ_i a random observation obeying a (possibly) unknown distribution function F with finite, but unknown, variance σ^2. We wish to find a region R in p-dimensional Euclidean space such that $P(\beta \varepsilon R) = 1 - \alpha$ and such that the length of the interval cut off on the β_i-axis by R has width $\leqq 2d$, $i = 1, \cdots p$. As has already been noted for $p = 1$, no fixed-sample procedure will meet our requirements; we are thereby led to consider sequential procedures.

To motivate the sequential procedure that we use, consider what classical statistical practice would be if σ^2 were known. Since the least-squares estimate of β has componentwise (by the Gauss-Markov theorem) uniformly minimum variance among all linear unbiased estimates of β, has good asymptotic properties (such as consistency—viz., Eicker [5]), and performs reasonably well against nonlinear unbiased estimates (Anderson [1]), classical practice would be to use the least-squares estimate of β in the construction of our confidence region. It is well-known that the least-squares estimate of β in our problem is

$$\hat{\beta}(n) = Y_n X_n'(X_n X_n')^{-1} \tag{2.2}$$

where $Y_n = (y_1, \cdots, y_n)$, $X_n = (x^{(1)}, \cdots, x^{(n)}) : p \times n$, $p \leqq n$, and where we assume that X_p is of full rank. (This is usually possible to achieve in practice—

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