THE EXPECTED NUMBER OF ZEROS OF A STATIONARY GAUSSIAN PROCESS[1]

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1. Introduction. It is assumed throughout that $\{X(t), t \varepsilon [0, 1]\}$ is a real separable stationary Gaussian process with mean function zero, covariance function ρ, $\rho(0) = 1$, and having continuous sample paths. It follows that there is a normalized (spectral) distribution function F such that $F(u) = 1 - F(-u)$ at points of continuity u and for which

$$EX(s)X(t) = \rho(s - t) = \int_{-\infty}^{\infty} \cos (s - t)\lambda \, dF(\lambda), \qquad s, t \varepsilon [0, 1].$$

We will use freely the equivalence of the two conditions: ρ'' exists continuous at the origin and $\int_{-\infty}^{\infty} \lambda^2 \, dF(\lambda) < \infty$; further, that these conditions imply $X(\cdot)$ is absolutely continuous a.s. (cf [3] p. 536).

Let N be the number of zeros of $X(\cdot)$. It is shown here that

(a) $X(\cdot)$ has a.s. no tangential zeros,

(b) $EN = (1/\pi)(-\rho''(0))^{\frac{1}{2}}$ if $\rho''(0)$ exists,
$= +\infty$ if not.

The formula for EN goes back to Rice [8] who obtains it under the assumption that F has finitely many points of increase. Ivanov [5] proves that EN is given above when $\rho^{(iv)}(0)$ exists (equivalently $\int_{-\infty}^{\infty} \lambda^4 \, dF(\lambda) < \infty$), while Bulinskaya [2] shows that (a) and (b) hold provided $X(\cdot)$ has a continuous derivative (for which the best sufficient condition known is $\int_{-\infty}^{\infty} \lambda^2 \log (1 + |\lambda|)^{1+\delta} \, dF(\lambda) < \infty$, $\delta > 0$). The result given here however is not unexpected (see for example [4], p. 273).

Most previous work in this area has followed Kac who in [6] devised a procedure for counting zeros; inasmuch as this procedure is designed to account for tangencies when there are none, we adopt a different, and perhaps simpler, approach to the counting. A suitable modification of what we do will produce the expected number of a's, $a \neq 0$, for the same processes and indications are that it can also be extended to handle zeros of nonstationary Gaussian processes with nonzero mean functions. Leadbetter and Cryer [7] have found the corresponding formula, applicable when the process has a continuous derivative, and they point out that this is the general situation for arbitrary barriers and Gaussian processes. For corresponding nonnormal problems it will be seen that the main calculations involve the bivariate distributions of the process with tangencies requiring special attention.

Received 28 October 1964.

[1] Research supported in part by the National Science Foundation under grants G-19645 and G-25211.