QUANTILES AND MEDIANS

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1. Introduction. The role played by the quantiles and in particular by the median of a random variable in probability theory and in mathematical statistics is well known. It is difficult to work with them, because they are not linear functionals of the random variables, even when these are independent. While the elementary properties of the expectation may be found in every textbook, the analogous properties are known neither for the quantiles nor for the median. This paper contains a contribution to this problem.

2. Definitions and notations. For every number $p(0 \leqq p \leqq 1; q = 1 - p)$ we define the *p-quantile* of the real valued random variable ξ as the real number $r(\xi; p) = r\xi$ for which the two inequalities

$$P\{\xi \leqq r\,\xi\} \geqq p, \qquad P\{\xi \geqq r\,\xi\} \geqq q \tag{1}$$

are simultaneously satisfied. The 0.5-quantile of ξ is his *median*, $m\xi$.

We denote by $M\xi$, $D\xi$ the expectation and the variance; by ξ^-, ξ^+ the infimum and the supremum of ξ.

Let us consider n random variables $\xi_i (1 \leqq i \leqq n)$; we denote

$$\begin{aligned}
&\sigma_n = \textstyle\sum_{i=1}^n \xi_i, \qquad \sigma_{1n} = \sum_{i=1}^n \xi_i^+, \qquad \sigma_{2n} = \sum_{i=1}^n \xi_i^-,\\
&S_n = r(\sigma_n; p) = r\sigma_n, \qquad S_n' = \textstyle\sum_{i=1}^n r(\xi_i; p) = \sum_{i=1}^n r\xi_i,\\
&\pi_n = \textstyle\prod_{i=1}^n \xi_i, \qquad P_n = r\pi_n, \qquad P_n' = \prod_{i=1}^n r\xi_i, \qquad D_n = \sum_{i=1}^n D\xi_i,\\
&U_i = S_i - (S_{i-1} + r\xi_i), \qquad \gamma = 9 + 6^{\frac{1}{2}}8, \qquad Q_1 = 2p^{-\frac{1}{2}} + 2^{\frac{1}{2}}q^{-\frac{1}{2}},\\
&Q_2 = 2^{\frac{1}{2}}p^{-\frac{1}{2}} + 2q^{-\frac{1}{2}}, \qquad Q_3 = 2p^{-\frac{1}{2}} + q^{-\frac{1}{2}}, \qquad Q_4 = p^{-\frac{1}{2}} + 2q^{-\frac{1}{2}}\\
&G_n^{(1)} = (n-1)^{\frac{1}{2}}Q_1, \qquad G_n^{(2)} = \gamma^{\frac{1}{2}}[\alpha^{(n-1)}]^{-\frac{1}{2}}Q_1, \qquad G_n^{(3)} = \gamma^{\frac{1}{2}}\rho^{-\frac{1}{2}}Q_1\\
&G_n^{(4)} = Q_3, \qquad H_n^{(1)} = -(n-1)^{\frac{1}{2}}Q_2, \qquad H_n^{(2)} = -\gamma^{\frac{1}{2}}[\alpha^{(n-1)}]^{-\frac{1}{2}}Q_2,\\
&H_n^{(3)} = -\gamma^{\frac{1}{2}}\rho^{-\frac{1}{2}}Q_2, \qquad H_n^{(4)} = Q_4, \qquad R_n^{(k)} = (n-1)G_n^{(k)},\\
&T_n^{(k)} = (n-1)H_n^{(k)} (1 \leqq k \leqq 4); \qquad I = (1, 2, 3, \cdots).
\end{aligned} \tag{2}$$

In the case where the sequence of random variables $\xi_i(1 \leqq i \leqq n)$ is a Markov chain, we denote by $\alpha_i(1 \leqq i \leqq n)$ the ergodic coefficient ([1], [2], [3], [4]) of its ith stochastic transition function. Let us denote

$$\alpha^{(n)} = \min_{1 \leqq i < n} \alpha_i . \tag{3}$$

The reader is referred to [3] for a survey of the definition and properties of α_i.

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