MOMENTS OF RANDOMLY STOPPED SUMS

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1. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a probability space, let $x_1, x_2, \cdots$ be a sequence of random variables on Ω, and let $\mathfrak{F}_n$ be the σ-algebra generated by $x_1, \cdots, x_n$, with $\mathfrak{F}_0 = (\phi, \Omega)$. A *stopping variable* (of the sequence $x_1, x_2, \cdots$) is a random variable t on Ω with positive integer values such that the event $[t = n] \varepsilon \mathfrak{F}_n$ for every $n \geqq 1$. Let $S_n = \sum_1^n x_i$; then $S_t = S_{t(\omega)}(\omega) = \sum_1^t x_i$ is a randomly stopped sum. *We shall always assume that*

$$(1) \qquad E|x_n| < \infty, \qquad E(x_{n+1} \mid \mathfrak{F}_n) = 0, \qquad (n \geqq 1).$$

The moments of S_t have been investigated since the advent of Sequential Analysis, beginning with Wald [9], whose theorem states that for *independent, identically distributed* (iid) x_i *with* $Ex_i = 0$, $Et < \infty$ implies that $ES_t = 0$. For higher moments of S_t, the known results [1, 3, 4, 5, 10] are not entirely satisfactory. We shall obtain theorems for ES_t^r $(r = 2, 3, 4)$; the case $r = 2$ is of special interest in applications. For iid x_i with $Ex_i = 0$ and $Ex_i^2 = \sigma^2 < \infty$, we shall show that $Et < \infty$ implies $ES_t^2 = \sigma^2 Et$.

2. The second moment. It follows from assumption (1) that $(S_n, \mathfrak{F}_n; n \geqq 1)$ is a *martingale*; i.e., that

$$(2) \qquad E|S_n| < \infty, \qquad E(S_{n+1} \mid \mathfrak{F}_n) = S_n \qquad (n \geqq 1).$$

The following well-known fact ([3], p. 302) will be stated as

LEMMA 1. *Let* $(S_n, \mathfrak{F}_n; n \geqq 1)$ *be a martingale and let* t *be any stopping variable such that*

$$(3) \qquad E|S_t| < \infty, \qquad \liminf \int_{[t>n]} |S_n| = 0;$$

then

$$(4) \qquad E(S_t \mid \mathfrak{F}_n) = S_n \quad \textit{if} \quad t \geqq n \qquad (n \geqq 1),$$

and hence $ES_t = ES_1$.

LEMMA 2. *If* $E\sum_1^t |x_i| < \infty$, *then* (3) *holds.*

PROOF. $|S_t| \leqq \sum_1^t |x_i|$, so that $E|S_t| < \infty$, and

$$\lim \int_{[t>n]} |S_n| \leqq \lim \int_{[t>n]} \sum_1^t |x_i| = 0.$$

In the remainder of this section we shall suppose, in addition to (1) that

$$(5) \qquad Ex_n^2 < \infty \qquad (n \geqq 1)$$

and we define for $n \geqq 1$

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