OPTIMAL INVARIANT RANK TESTS FOR THE k-SAMPLE PROBLEM[1]

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0. Introduction. Suppose that we have k random samples of size n from populations with distribution functions $F_1, F_2, \cdots, F_k$ all belonging to the same class Ω of distribution functions. That is, $X_1, X_2, \cdots, X_{kn}$ are independent random variables, and X_i has distribution function F_α if $(\alpha - 1)n < i \leqq \alpha n$. The k-sample problem is to test the hypothesis that $F_1 = F_2 = \cdots = F_k$. The hypothesis, as well as the alternative that the distribution functions are not identical, remains invariant under relabelling of the distribution functions, and it is natural to ask that a test also be invariant under a relabelling of the samples. In this paper we will consider only tests which are invariant under all permutations of the k samples.

In Section 1, one-parameter families of distributions are introduced, and the locally most powerful invariant rank tests are found. These tests are all based on a statistic of the form

$$\sum_{i=1}^{kn} \sum_{j=1}^{kn} (W_{ij} - \bar{W})a_{ij}, \tag{0.1}$$

where the a_{ij} are constants depending on the one-parameter family, and W_{ij} is a random variable which is one if the ith and the jth order statistic from the combined sample both come from the same sample, and $W_{ij} = 0$ otherwise. We take $W_{ii} = 1$ and $\bar{W} = 1/k$.

In Section 2 the limiting distribution of this statistic is found under the null hypothesis, and under a sequence of alternatives in Section 3. In Section 4 the locally best invariant rank test statistic is shown to be asymptotically equivalent to a quadratic form in certain statistics which arise in the two-sample problem. In Section 5 it is proved for many families of alternatives which include those of the translation type, that the locally best invariant rank test is asymptotically equivalent to the test which maximizes, among all tests, the average power over spheres in the parameter space. This optimality property is an asymptotic analogue of the well-known F test and seems not to have been previously discussed. In Section 6 the general results are applied in a special case showing that the Kruskal-Wallis test possesses the cited optimality properties.

The point of departure in the present paper is the investigation of invariant rank tests. Previously, locally most powerful rank tests against one-sided parametric alternatives $\theta > 0$ have been obtained. For the two-sample problem such tests reject the hypothesis $\theta = 0$ when $L = \sum b_i Z_i >$ constant, where $\{Z_i\}$ is the rank order vector, $Z_i = 1$, or 0, according as the ith ordered observation is, or is not, from the first sample, and where the b_i's are constants depending on the

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