SOME APPLICATIONS OF MONOTONE OPERATORS  
IN MARKOV PROCESSES

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1. Introduction. This paper establishes uniqueness, stability, and methods of error estimation for a broad class of integro-differential equations that arise in the study of Markov processes. In Section 2 we consider the equation $Tu = 0$, where $T$ is defined for real-valued differentiable functions $u$ on an interval $S$ of the real line by

$$
(Tu)(x) = u(x) - g(x, u'(x)) - \alpha(x) \int_S u(y) \, dF_x(y).
$$

Here $g(x, y)$ is a real-valued function defined for all $x \in S$ and all real $y$, $\alpha(x)$ satisfies $0 \leq \alpha(x) \leq 1$, and for each $x \in S$, $F_x$ is a distribution function on $S$. In Section 3 the treatment is extended to an arbitrary space $S$. Here the variable $u'$ is suppressed and $T$ is defined for real-valued functions $u$ on $S$ by

$$
(Tu)(x) = u(x) - g(x) - \alpha(x) \int_S u(y) \, dP_x(y),
$$

where for each $x \in S$, $P_x$ is a probability measure on a fixed $\sigma$-algebra in $S$.

Using very elementary methods, it is shown that these operators are monotone in the sense of Collatz [3], viz., $Tu \leq Tv \Rightarrow u \leq v$. The uniqueness, stability, and error estimation mentioned above are easily obtained from this property.

Equations of the type $Tu = 0$ are frequently satisfied by absorption probabilities, mean passage times and various other expectations associated with a Markov process in the space $S$. Some examples illustrating how the operator $T$ arises are described in Section 4. The same methods have been extended to the functional equations encountered in Markovian decision problems [1], [2]. These applications are considered elsewhere.

2. $S$ an interval of the real line. Let $S$ be a finite or infinite interval of the real line, and let $S^*$ be its open interior $(a, b)$. We say ‘$u \leq v$ on the boundary’ if $\lim_{x \to a^+} u(x) = v(x)$, and ‘$u = v$ on the boundary’ if $\lim_{x \to b^-} [u(x) - v(x)] = 0$.

We consider now the operator $T$ defined by (1). The distribution function $F_x$ is taken continuous from the right.

Theorem 1. Suppose $u \leq v$ on the boundary. If $\alpha(x)F_x(x) < 1$ at each $x \in S^*$, then $Tu \leq Tv$ on $S^*$ implies $u \leq v$.

Proof. Suppose $\sup_{x \in S} [u(x) - v(x)] = m > 0$, with $m$ necessarily finite. Then there is a largest value of $x$, say $x_0$, such that $u - v = m$ at $x_0$; moreover, $x_0 < b$, since $u \leq v$ on the boundary. To the right of $x_0$ we have $u - v < m$, at

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