SOME APPLICATIONS OF MONOTONE OPERATORS IN MARKOV PROCESSES

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1. Introduction. This paper establishes uniqueness, stability, and methods of error estimation for a broad class of integro-differential equations that arise in the study of Markov processes. In Section 2 we consider the equation $Tu = 0$, where T is defined for real-valued differentiable functions u on an interval S of the real line by

$$(1) \qquad (Tu)(x) = u(x) - g(x, u'(x)) - \alpha(x) \int_S u(y)\, dF_x(y).$$

Here $g(x, y)$ is a real-valued function defined for all $x \,\varepsilon\, S$ and all real y, $\alpha(x)$ satisfies $0 \leqq \alpha(x) \leqq 1$, and for each $x \,\varepsilon\, S$, F_x is a distribution function on S. In Section 3 the treatment is extended to an arbitrary space S. Here the variable u' is suppressed and T is defined for real-valued functions u on S by

$$(2) \qquad (Tu)(x) = u(x) - g(x) - \alpha(x) \int_S u(y)\, dP_x(y),$$

where for each $x \,\varepsilon\, S$, P_x is a probability measure on a fixed σ-algebra in S.

Using very elementary methods, it is shown that these operators are monotone in the sense of Collatz [3], *viz.*, $Tu \leqq Tv \Rightarrow u \leqq v$. The uniqueness, stability, and error estimation mentioned above are easily obtained from this property.

Equations of the type $Tu = 0$ are frequently satisfied by absorption probabilities, mean passage times and various other expectations associated with a Markov process in the space S. Some examples illustrating how the operator T arises are described in Section 4. The same methods have been extended to the functional equations encountered in Markovian decision problems [1], [2]. These applications are considered elsewhere.

2. S an interval of the real line. Let S be a finite or infinite interval of the real line, and let S^* be its open interior (a, b). We say '$u \leqq v$ on the boundary' if $\limsup_{x\to a+, x\to b-} [u(x) - v(x)] \leqq 0$, and '$u = v$ on the boundary' if $\lim_{x\to a+, x\to b-} [u(x) - v(x)] = 0$.

We consider now the operator T defined by (1). The distribution function F_x is taken continuous from the right.

Theorem 1. *Suppose $u \leqq v$ on the boundary. If $\alpha(x)F_x(x) < 1$ at each $x \,\varepsilon\, S^*$, then $Tu \leqq Tv$ on S^* implies $u \leqq v$.*

Proof. Suppose $\sup_x [u(x) - v(x)] = m > 0$, with m necessarily finite. Then there is a largest value of x, say x_0, such that $u - v = m$ at x_0; moreover, $x_0 < b$, since $u \leqq v$ on the boundary. To the right of x_0 we have $u - v < m$, at

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