THE LIMIT OF THE nth POWER OF A DENSITY[1]

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1. Introduction. If $f(\cdot)$ is a bounded density function of an absolutely continuous variate z, then the powers $f^2, f^3, \cdots$, can be normalized to define new variates $z_2, z_3, \cdots$. Typically, z_n will converge in probability to the mode (say m) of $f(\cdot)$, and it is shown below (Corollary 2) that if f is unimodal, $f'(m) = 0$, and $f''(m) \neq 0$, then $y_n = n^{\frac{1}{2}}(z_n - m)$ will tend in distribution to a normal variate with mean equal to zero and variance equal to $-f(m)/f''(m)$. Four examples of this result, relating to gamma, beta, Student's t, and Snedecor's F variates, are given in Section 3. Asymptotic normality is of course well known for these cases.

Our main result, Theorem 1, is more general than Corollary 2 in two respects:

(a) The density of z_n is assumed to have the form

$$c_n\{f(z)\}^n k(z) \tag{1}$$

where c_n is a constant and $k(z)$ is bounded. Order statistics have densities of this form, and their asymptotic normality is a consequence (Example 5).

(b) The conditions $f'(m) = 0, f''(m) \neq 0$, are relaxed to allow more general behavior at the mode. We allow cusps, as exemplified by $f(z) = 1 - |z|$, $(|z| < 1)$, or "flat" maxima for which $f''(m) = 0$. In these cases a limiting density is obtained having the form $c \exp\{-|y|^\gamma\}$ where γ is the order of the first nonvanishing term in the Taylor expansion of $f(z) - f(m)$.

Theorem 2 is a multivariate analog of Theorem 1, applicable for example to the Dirichlet distribution.

Theorem 1 is proved by first expanding f in its Taylor series about the mode, and taking the limit of the nth power after "standardizing" the variate. The result of this routine calculation is easily anticipated. The difficulty lies in the normalization constants. By truncating the densities and by appealing to the dominated convergence theorem, it is shown without evaluating the normalization constants that these constants converge as desired. Convergence in distribution is then established by Scheffé's theorem.

It may be instructive to cite an example wherein the assumptions are violated in such a way that Scheffé's theorem is inapplicable and the conclusion of Theorem 1 is false. Suppose $f(z)$ has a local maximum at $z = 0$ and an absolute maximum at $z = 1$, with, say, $f(0) = 1, f''(0) = -1, f(1) = 2$. If z_n has density proportional to f^n, then certain constant multiples of the densities of $y_n = (2n)^{\frac{1}{2}} z_n$ will approach $\exp\{-y^2/2\}$; but the densities themselves would everywhere ap-

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