ON AN OPERATOR LIMIT THEOREM OF ROTA

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1. Introduction. Let (X, Σ, μ) be a probability measure space and let $f \varepsilon L^p(X, \Sigma, \mu)$ for some $p > 1$. Given a sequence $P_1, P_2, \cdots$ of doubly stochastic operators on $L^1(X, \Sigma, \mu)$, Rota [4] has shown that $\lim_{n\to\infty} P_1{}^*P_2{}^* \cdots P_n{}^*P_nP_{n-1} \cdots P_1f$ exists a.e. (Convergence in the L^p norm also holds, by a variety of proofs $(1 \leqq p < \infty)$. Almost everywhere convergence does not extend to the case $p = 1$ [1].) In the same article it was stated that a convergence theorem using the reverse index product $P_n{}^*P_{n-1}^* \cdots P_1{}^*P_1P_2 \cdots P_n$ was a possible generalization. The impossibility of such a result, or even of a strongly continuous inhomogeneous semi-group analog thereof, is shown in this note. We thank D. L. Burkholder for permission to include, as a second example, a result he discovered in connection with a distinct problem.

By a *doubly stochastic operator* on $L^1(X, \Sigma, \mu)$ is meant a linear operator on $L^1(X, \Sigma, \mu)$ into itself such that

(1) $\int |Pf| \, d\mu \leqq \int |f| \, d\mu$,

(2) $f \geqq 0$ a.e. $\Rightarrow Pf \geqq 0$ a.e., and

(3) $P1 = 1$ a.e., where 1 is the constant function assuming everywhere the value 1.

It is readily shown that the adjoint operator P^* satisfies (2) and (3) and does not increase L^1 norms of L^∞ functions. Thus P^* may be extended to an operator on L^1 and this extension, denoted also P^*, is doubly stochastic.

We say a family $\{P(t, s)\}_{\{t \geqq s \geqq 0\}}$ of bounded operators on a Banach space B is an *inhomogeneous semi-group* if $P(t, r)P(r, s) = P(t, s)$ for $t \geqq r \geqq s$ and $P(t, t) = I$, the identity operator. The semi-group is *strongly continuous* if for each $b \varepsilon B$, $P(t, s)b$ is a continuous function on the subset $\{t \geqq s \geqq 0\}$ of R^2 to B.

Given $f \varepsilon L^1(X, \Sigma, \mu)$, $E\{\cdot \mid f\}$ denotes conditional expectation with respect to the σ-field determined by f.

2. Continuous parameter example. Let $\{P(t, s)\}$ be an inhomogeneous semi-group of doubly stochastic operators on $L^1(X, \Sigma, \mu)$. Using the separability theory of Doob [2], the following version of Rota's theorem can be proved: Given $p > 1$ and $f \varepsilon L^p(X, \Sigma, \mu)$, the family $\{P^*(t, 0)P(t, 0)f\}_{t\varepsilon[0,\infty)}$ can be redefined for each t on a set of μ-measure zero in such a manner that $\lim_{t\to\infty} P^*(t, 0)P(t, 0)f$ exists everywhere. If one reverses the operations and considers the limiting behavior of $P(t, 0)P^*(t, 0)f$ as $t \to \infty$, pointwise convergence need not hold, even if $\{P(t, s)\}$ is *strongly continuous* in each L^p $(1 \leqq p < \infty)$. This we now show.

Let (X, Σ, μ) be the unit circle with normalized Lebesgue measure: $d\mu =$

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