# A UNIFORM ERGODIC THEOREM[1]

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**1. Introduction.** Let $\{X_n\}$ be a sequence of independent random variables with common distribution function $F(t)$, and let $F_n(t, \omega)$ be the $n$th empirical distribution function of the sequence. Then the Glivenko-Cantelli theorem ([3], p. 20), states that for almost all $\omega$, $F_n(t, \omega)$ converges to $F(t)$ uniformly in $t$. In [4], Tucker has shown that even if $\{X_n\}$ is only strictly stationary $F_n(t, \omega)$ is still uniformly convergent for almost all $\omega$, the limit being $F(t, \omega \mid \mathfrak{g})$, the conditional distribution function of $X_1$ given $\mathfrak{g}$, the invariant field of the sequence. Another generalization of the Glivenko-Cantelli theorem was accomplished by Fisz [2], who noted that for each fixed $n$, $F_n(t, \omega)$ could be looked upon as a non-decreasing stochastic process and for each fixed $t$ the sequence of arithmetic means derived from a sequence of independent random variables.

Looking at Tucker's theorem in this light, we could rephrase it as follows. Let $X$ be a random variable, let $X(t) = I_{\{X \leqq t\}}$, and let $T$ be a measure preserving set transformation. Choose $X_k(t) = T^k(X(t))$ in such a way that for each $k$, $X_k(t)$ is non-decreasing and right continuous. Then for almost all $\omega$, $n^{-1}\sum_{k=1}^{n} X_k(t, \omega)$ converges uniformly in $t$. It is our purpose in this paper to show that this result remains true whenever $X(t)$ is any non-decreasing, right continuous process with $E(X(t))$ bounded. The proof is based on a general criterion for uniform convergence of a sequence of monotone processes and some results on conditional expectations which may prove of interest in themselves.

**2. Conditional expectations for non-decreasing right continuous processes.** Let $X(t, \omega)$ be a non-decreasing, right continuous process where $t$ ranges over all real numbers. Let $Y$ be an extended real valued function defined on our probability space $\Omega$. We define

$$X(Y)(\omega) = X(Y(\omega), \omega).$$

For each real $a$, we define

$$Y_a(\omega) = \inf \{t: X(t, \omega) \geqq a\}.$$

The following result is then easily seen.

THEOREM 1. *For each real $a$ and $t$, $\{Y_a > t\} = \{X(t) < a\}$ and $\{X(Y) < a\} = \{Y < Y_a\}$.*

In view of Theorem 1 it is obvious that $Y_a$ is always $\mathfrak{F}(X(t): -\infty < t < \infty)$-measurable, and that $X(Y)$ is always measurable whenever $Y$ is measurable.

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