CYLINDRICALLY ROTATABLE DESIGNS[1]

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1. Introduction. In what follows we use k-dimensional space to describe an experimental design for k factors. We refer to the combinations of levels of factors used in the design as points in k-dimensional space.

Box and Hunter (1957) gave conditions under which designs for the fitting of response surfaces would be rotatable. (Response surface designs are said to be rotatable if the variances of the estimated responses at all points equidistant from the origin of the design are equal.) These conditions for rotatability are restrictive. Some methods for forming rotatable designs are given in Bose and Draper (1959) and Draper (1960a, b). It is desirable to find designs which are both in some sense rotatable and also practical for the experimenter to employ.

Here we consider designs such that the variances of the estimated responses at points on the same $(k - 1)$-dimensional hyper-sphere centred on a specified axis are equal. We shall call such designs *cylindrically rotatable designs*. If the experimental design is rotated about the specified axis, the variances and covariances of the estimated coefficients of the response function remain unchanged. A cylindrically rotatable design is identical to a rotatable design of the same order except in the required levels of one factor.

2. Conditions for cylindrically rotatable designs. We assume that there are k factors whose standardized levels are denoted by $x_1, x_2, \cdots, x_k$. We also assume that the response surface may be represented in a given region by a polynomial of degree d, i.e.

$$\eta(\mathbf{x}) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k + \beta_{11} x_1^2 + \cdots + \beta_{kk} x_k^2 + \beta_{12} x_1 x_2 + \cdots + \beta_{k-1,k} x_{k-1} x_k + \beta_{111} x_1^3 + \cdots, \qquad \text{of degree } d.$$

Let $\hat{y}(\mathbf{x})$ denote the estimated response at $\mathbf{x}$, where $\mathbf{x} = (x_1, x_2, \cdots, x_k)$. $\hat{y}(\mathbf{x})$ is the polynomial fitted by least squares to the response surface from the N observations made according to some particular design. Let $V(\hat{y}(\mathbf{x}))$ be the variance of the estimated value of the response at $\mathbf{x}$. We want to find conditions such that the variances of the estimated responses at all points on the same $(k - 1)$-dimensional hyper-sphere centered on the axis $x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_k = 0$ are equal.

Suppose

$$(1) \quad \hat{y}(\mathbf{x}) = b_0 + b_1 x_1 + \cdots + b_k x_k + b_{11} x_1^2 + \cdots + b_{kk} x_k^2 + b_{12} x_1 x_2 + \cdots + b_{k-1,k} x_{k-1} x_k + b_{111} x_1^3 + \cdots$$

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