THE PERFORMANCE OF A SEQUENTIAL PROCEDURE FOR THE FIXED-WIDTH INTERVAL ESTIMATION OF THE MEAN[1]

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1. The problem. Let

$$X_1, X_2, \cdots \tag{1}$$

be independent $N(\theta, \sigma^2)$ with $\sigma^2 < \infty$. Suppose we require a confidence interval for θ of width at most $2d$ $(d > 0)$ and with probability of coverage at least α $(0 < \alpha < 1)$, irrespective of the values of θ and σ^2. Define for $n \geqq 2$

$$\bar{X}_n = n^{-1} \textstyle\sum_{i=1}^n X_i, \qquad S_n^2 = (n-1)^{-1} \textstyle\sum_{i=1}^n (X_i - \bar{X}_n)^2,$$

and for $x > 0$ let

$$\varphi(x) = (2\pi)^{-\frac{1}{2}} \textstyle\int_{-x}^{x} e^{-t^2/2}\, dt;$$

a = constant for which $\varphi(a) = \alpha$, and

$$\varphi_n(x) = \Gamma((n+1)/2)/(n\pi)^{\frac{1}{2}}\Gamma(n/2) \textstyle\int_{-x}^{x} (1 + t^2/2)^{-(n+1)/2}\, dt; \tag{2}$$

a_n = constant for which $\varphi_n(a_n) = \alpha$ $(n = 1, 2, \cdots)$; then

$$\lim_{n\to\infty} a_n = a. \tag{3}$$

Observe that if σ is *known* a confidence interval $I_n = [\bar{X}_n - d, \bar{X}_n + d]$ for θ of width $2d$ and with coverage probability $\geqq \alpha$ is assured provided n is chosen so that

$$n \geqq a^2\sigma^2/d^2, \tag{4}$$

since then $P(\theta \varepsilon I_n) = \varphi(n^{\frac{1}{2}} d/\sigma) \geqq \alpha$. However, it is clear that no procedure based on a fixed number n of observations of (1) satisfies the requirements when σ is *unknown*. In this circumstance one recourse is to two-stage sampling [10]. The Stein procedure leads to an n which approximately satisfies (4) with σ^2 estimated from the initial sample of size n_1 (and a increased to a_{n_1} to reflect the limited degrees of freedom). It seems intuitively inefficient not to utilize all of the sample; we shall investigate the performance of a sequential procedure Λ which does just this, leading to an n satisfying (4) with σ^2 estimated on $n - 1$ degrees of freedom. Accordingly, we prescribe the rule

Λ: Observe the sequence (1) term by term, stopping with X_N, where

$$N \text{ is the first integer } n \geqq n_0 \text{ such that } S_n^2 \leqq n\, d^2/a_{n-1}^2, \tag{5}$$

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