A NOTE ON LIMIT THEOREMS FOR THE ENTROPY OF MARKOV CHAINS[1]

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Let X_j, $j = 1, 2, \cdots$ be a stationary ergodic m-step Markov chain defined on a probability space (Ω, Q, P) and having for its state space the finite set of integers $\{0, 1, \cdots, D - 1\}$. Here Ω, the sample (path) space, is equal to the set of all sequences $(\omega_1, \omega_2, \cdots)$ with $\omega_n = X_n(\omega) \varepsilon \{0, 1, \cdots, D - 1\}$. Q is the Borel field generated by the cylinder sets of Ω and P is a stationary probability measure on Ω which for all $n \geqq m + 1$ satisfies the relation

$$P\{X_n = i_n/X_{n-1} = i_{n-1}, \cdots, X_1 = i_1\} = P\{X_{m+1} = i_n/X_m = i_{n-1}, \cdots, X_1 = i_{n-m}\}$$

where $i_K \varepsilon \{0, 1, \cdots, D - 1\}$, $K = 1, \cdots, n$.

For $\omega \varepsilon \Omega$ let $[\omega]_n$ denote the cylinder set $\{u \varepsilon \Omega: u_1 = \omega_1, \cdots, u_n = \omega_n\}$ and correspondingly let $P([\cdot]_n)$ denote the random variable whose value at ω is $P([\omega]_n) = P\{u \varepsilon \Omega: u \varepsilon [\omega]_n\}$. In this note we establish a law of the iterated logarithm for the sequence of random variables $\{-\log P([\cdot]_n)\}$:

THEOREM 1.

$$P\{\omega: \limsup_{n\to\infty} [(-\log P([\omega]_n) - nH)/(2Bn \log \log n)^{\frac{1}{2}}] = 1\} = 1$$

where H denotes the entropy rate of the process X_n, i.e.,

$$H = \lim_{n\to\infty} [E(-\log P([\omega]_n))/n]$$

and

$$B = \lim_{n\to\infty} [E\{(-\log P([\omega]_n) - nH)^2\}/n].$$

E denotes the expectation operator relative to the measure P.

The proof of Theorem 1, to be presented below, depends essentially upon the observation that there exists a function f and a one-step Markov chain $Z_j(\omega)$, $j = m + 1, m + 2, \cdots, \omega \varepsilon \Omega$, such that

$$-\log P([\omega]_n) = \textstyle\sum_{j=m+1}^{n} f(Z_j(\omega)) + O(1). \qquad (1)$$

The proof is then completed by referring to a version of the law of the iterated logarithm theorem which is applicable to functionals of a Markov chain (Chung, [3], Theorem 5, p. 101). It is worth noting that (1) may be used in conjunction with other established limit theorems for functionals of a Markov chain, (Chung,

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