ON THE MEAN DURATION OF A BALL AND CELL GAME; A FIRST PASSAGE PROBLEM[1]

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1. Introduction. In this paper we study the mean duration of the following r ball n cell game:

Each of r balls is placed at random into one of n cells. A ball is considered "captured" if (after all r balls have been distributed) it is the sole occupant of its cell. Captured balls are eliminated from further play. This completes the first "trial." The remaining balls are recovered and the process repeated (trials, 2, 3, 4, $\cdots$ etc.). The play continues until all balls have been captured. The number of trials required to achieve this state is called the duration of the game.

We first show, in Section 2, that the probability of exactly $r - t$ balls remaining in play (or of exactly t balls being "captured") is equal to

$$P_{r,r-t}(n) = \sum_{j=t}^{n} (-1)^{j-t}\binom{j}{t}\binom{n}{j}\binom{r}{j}j!\,[(n-j)^{r-j}/n^r].$$

Various bounds on the probability of this event are then derived for subsequent use. When the intent is clear the n dependency may be suppressed and the symbol P_{rt} used instead of $P_{rt}(n)$. The notation is suggestive of that used in the theory of Markov chains. Indeed, the r ball n cell game may be identified with an $r + 1$ state Markov chain, the states being the number of balls possibly in play at any stage: r, $r - 1$, $\cdots$, 2, 1, or 0. In keeping with conventional Markov chain terminology we shall refer to the quantities $P_{rt}(n)$ as transition probabilities. Note also that the mean duration of an r ball game is equal to the mean first passage time to state "0" from initial state "r."

Denoting the mean duration of the game by $M_n(r)$ (or simply $M(r)$ if there is no ambiguity) we proceed in Section 3 to derive the bounds:

$$r^{-1}[n/(n-1)]^{r-1} \leqq M_n(r) \leqq \sum_{j=1}^{r} [1 - P_{jj}(n)]^{-1} \leqq n^2[n/(n-1)]^{r-1}.$$

The principal result of this paper, namely that

$$M_n(r) = \sum_{j=1}^{r} j^{-1}[n/(n-1)]^{j-1} + O(1) \qquad (r \to \infty, n \quad \text{fixed})$$

appears in Section 4. There it is also shown that

$$M_n(r) = \sum_{j=1}^{r} [1 - P_{jj}(n)]^{-1} + O(1) \qquad (r \to \infty, n \quad \text{fixed}).$$

The arguments leading to this last result which appear in Section 4 (up to and including Theorem 1) are presented in a form suggested by the referee. They

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