A FAMILY OF COMBINATORIAL IDENTITIES

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Given an ordered n-tuple of real numbers, $(x_1, x_2, \cdots, x_n)$, let σ denote any cyclic permutation of these numbers. If $\sigma = (y_1, \cdots, y_n)$ then σ_j denotes the ordered n-tuple defined by $\sigma_j = (y_j, y_{j-1}, \cdots, y_1, y_n, y_{n-1}, \cdots, y_{j+1})$. In particular, if $\sigma = (y_1, \cdots, y_n)$, then $\sigma_n = (y_n, \cdots, y_1)$. Note that $(\sigma_j)_j = \sigma$, so that the operation is 1-1 and onto.

Now we make the following definition which extends the notation used by R. L. Graham in [3].

DEFINITION 1. $M_{rj}(z_1, \cdots, z_n)$ and $m_{rj}(z_1, \cdots, z_n)$ denote the rth largest and the rth smallest, respectively, among the first j partial sums $z_1, z_1 + z_2, \cdots, z_1 + \cdots + z_j$ for $1 \leqq r \leqq j \leqq n$. Note that $M_{rj} = m_{j-r+1,j}$.

DEFINITION 2. If x is a real number, then

$$x^+ = x \quad \text{if} \quad x \geqq 0$$
$$= 0 \quad \text{if} \quad x < 0$$

and

$$x^- = 0 \quad \text{if} \quad x \geqq 0$$
$$= x \quad \text{if} \quad x < 0.$$

Note that $x = x^+ + x^-$.

THEOREM.

$$\sum_\sigma [M^+_{rj}(\sigma) + m^-_{rj}(\sigma_n)] = (j - r + 1)s_n \tag{1}$$

where the sum is taken over all cyclic permutations of $(x_1, \cdots, x_n)$, *a total of* n, *and* $s_n = x_1 + \cdots + x_n$.

This formula, the main result of the note, is a generalization of a combinatorial theorem on partial sums by R. L. Graham [3], which appears here as Corollary 2. Graham had generalized a result of M. Dwass [1] and our extension includes another formula of Dwass' from the same paper, here Corollary 1.

PROOF. The proof is based upon two identities:

$$M^+_{rj}(\sigma) + m^-_{rj}(\sigma_j) = M^+_{r-1,j-1}(\sigma) + m^-_{r-1,j-1}(\sigma_j) \tag{2}$$

and

$$M^+_{1j}(\sigma) + m^-_{1j}(\sigma_j) = s_j, \tag{3}$$

where $\sigma = (y_1, \cdots, y_n)$ and $s_j = y_1 + \cdots + y_j$ for $1 \leqq r \leqq j \leqq n$. The intuitive idea behind these identities is a geometrical one, a variant of the reflection prin-

Received 19 April 1965; revised 10 November 1965.