INFINITELY DIFFERENTIABLE POSITIVE DEFINITE FUNCTIONS[1]

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The class of functions which are continuous and positive definite on the real line will be denoted by P. This paper presents two types of results concerning the derivatives of functions in P. The first type of result is that if a function in P agrees with a comparison function having certain properties on a sequence tending to the origin then the positive definite function either is identical to the comparison function or at least shares some of its properties. In 1960 R. G. Laha [6] proved the following theorem of this type:

A. Suppose $f \varepsilon P$ and g is analytic on the line (that is, g is the restriction to the real line of a function analytic in a strip of the complex plane along the line). If f and g agree on a double sequence x_k, $k = \pm 1, \pm 2, \cdots$, where $x_{-k} = -x_k$ and $x_k \to 0$, then f is analytic. Therefore $f = g$ on the line. Earlier, in a mimeographed note A. Devinatz [4] gave a similar result:

B. Suppose $f \varepsilon P$, $g \varepsilon C^\infty \cap P$, and the Hamburger moment sequence $(-i)^n g^{(n)}(0)$ is determined. If f and g agree on a sequence tending to the origin then $f = g$.

The second type of result is that certain properties possessed by a product of functions in P are shared by the factors. For instance, if the product of functions in P is infinitely differentiable or analytic then so are the factors. In 1959 A. Devinatz [5] proved the following result of this type:

C. Suppose $g \varepsilon P$, $h \varepsilon P$ and $f = gh$ is $2n$ times differentiable. Then g and h are also $2n$ times differentiable. For real r put $F_r(x) = e^{irx}f(x)$. Then for some real r, $|g^{(2k)}(0)| \leqq 2\,|F_r^{(2k)}(0)|$, $k = 0, \cdots, n$. A similar inequality holds for h.

We shall state our results in terms of certain classes of infinitely differentiable functions which were introduced by T. Carleman and S. Mandelbrojt. A positive sequence m_n is said to be *logarithmically convex* when the sequence $\log m_n$ is convex. A more useful equivalent definition of logarithmic convexity is that

$$m_0/m_1 \geqq m_1/m_2 \geqq \cdots \geqq m_n/m_{n+1} \geqq \cdots. \tag{1}$$

For a logarithmically convex sequence m_n we denote by $C(m_n)$ the class of functions, infinitely differentiable on the line, for which $f \varepsilon C(m_n)$ means that there is a finite $q = q(f)$ such that

$$\sup_x |f^{(n)}(x)| < q^n m_n, \qquad n = 0, 1, \cdots.$$

The purpose of introducing these classes of functions is to generalize certain

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