SOME EQUIVALENCE CLASSES IN PAIRED COMPARISONS[1]

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Introduction. In a paired comparison experiment n judges give a preference in some or all of the $\binom{t}{2}$ pairs of t items. Frequently the purpose of the experiment is to test the null hypothesis that every preference is equally likely against a vaguely defined alternative of consistency. Our purpose is to study several of the tests used, from the point of view of a natural equivalence relation which arises in graph theory. In the first section we introduce graph theory notation, the equivalence relation, and some results on partial and strict orderings on the equivalence classes. The succeeding section applies these notions to Kendall and Babington Smith's statistic in detail (hereafter simply referred to as Kendall's statistic), and mentions applications in the Bradley-Terry model, and the strong-stochastic ordering model.

1. Notions from graph theory. We define a paired comparison experiment, for these purposes, to consist of

(i) a set X of t items, which are the items being compared by the judges, and

(ii) n ordered relations R_k ($k = 1, \cdots, n$), subsets of $X \times X$, which are the preferences of the n judges. Thus $(x_i, x_j) \varepsilon R_k$ is interpreted to mean that item x_i is preferred to item x_j by the kth judge. We require that these n ordered relations be

(a) *anti-symmetric* $[(x, y) \varepsilon R_k \Rightarrow (y, x) \not\varepsilon R_k]$, thus each pair is judged at most once.

(b) *anti-reflexive* $[(x, x) \not\varepsilon R_k]$. No item is to be thought of as being preferred to itself.

A *path* K in $(R_1, \cdots, R_n) = R$ from y_1 to y_k, denoted $(y_1, \cdots, y_k)$ is a finite collection of ordered pairs $(y_1, y_2) \varepsilon R_{i_1}, \cdots, (y_{k-1}, y_k) \varepsilon R_{i_{k-1}}$. If $y_1 = y_k$, the path is called a *circuit.* If x and y are in some circuit together or $x = y$ then x and y are said to be *equivalent* (written $x \equiv y$). It is immediate that $\equiv$ is an equivalence relation. If $(x, y) \varepsilon R$ but x and y are not equivalent, then we may say x is an *ancestor* of y, or y is a *descendant* of x.

We will now study a natural ordering on the equivalence classes of the above equivalence relation.

Theorem 1. *There is a natural partial ordering on the equivalence classes. This ordering is given by*

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