LINEAR COMBINATIONS OF NON-CENTRAL CHI-SQUARE VARIATES[1]

By Sheldon James Press[2]

The RAND Corporation, Santa Monica, California

1. Introduction. Let $\chi^2_{m,d}$ denote a non-central chi-square variate with m degrees of freedom and non-centrality parameter d, whose probability density function is given by

$$p(x) = [x^{(m-2)/2}/2^{m/2}(\pi)^{\frac{1}{2}}]e^{-(d+x)/2}\sum_{j=0}^{\infty}[(x\,d)^j\Gamma(j + \tfrac{1}{2})/(2j)!\,\Gamma(j + (m/2))],$$

for $x > 0$, and zero otherwise. Define

$$T = U - V, \tag{1.1}$$

$$U = \alpha[\chi^2_{m_0,d_0} + \sum_{i=1}^{r} a_i\chi^2_{m_i,d_i}], \tag{1.2}$$

$$V = \beta[\chi^2_{n_0,g_0} + \sum_{j=1}^{s} b_j\chi^2_{n_j,g_j}], \tag{1.3}$$

where $\alpha > 0$, $\beta > 0$, $a_i \geqq 1$, $b_j \geqq 1$, $d_i \geqq 0$, $d_0 \geqq 0$, $g_j \geqq 0$, $g_0 \geqq 0$, for all i and j, and all chi-square variates are independent. The problem considered in this paper is the determination of the distribution of T for fixed known values of the parameters. Clearly if $\sum_1^p \lambda_i w_i^2$ is an arbitrary quadratic form in the w_i's in which the constants λ_i are real numbers, and the w_i's are independent random variables each of which has a normal distribution with non-zero mean and unit variance, by a minor bookkeeping change in notation, the form can be given the representation of (1.1)–(1.3). Hence, we are equivalently concerned with indefinite quadratic forms in non-central normal variates. Note that U and V are each expressible as some positive definite quadratic form.

Development of the distribution for an indefinite quadratic form in non-central normal variates was motivated by a classification problem. Consider the problem of classifying an unknown vector observation into one of two multivariate normal populations which have unequal means and covariance matrices. It may be shown (see [5]) that likelihood ratio procedures for this problem lead to consideration of the distribution developed in this paper.

Section 2 will be devoted to the distribution of positive definite quadratic forms in non-central normal variates.

In Section 3, the probability density function of T is found in terms of the results developed in Section 2.

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