ON THE MOMENTS OF SOME ONE-SIDED STOPPING RULES[1]

By Y. S. Chow

Columbia University and Purdue University

1. Introduction. The moments of stopping rules (or stopping times) have been discussed in [1], [3], and [4], and the following results have been proved. Let x_n be independent random variables with $Ex_n = 0$, $Ex_n^2 = 1$, and $S_n = x_1 + \cdots + x_n$. For $c > 0$ and $m = 1, 2, \cdots$, define t_m to be the first $n \geqq m$ such that $|S_n| > cn^{\frac{1}{2}}$. If $c \geqq 1$, then $Et_1 = \infty$. If $P[|x_n| \leqq K] = 1$ for some $K < \infty$ and $n = 1, 2, \cdots$, then $Et_m < \infty$ for every m if $c < 1$, $Et_m^2 < \infty$ for every m if $c < 3 - 6^{\frac{1}{2}}$, and $Et_m^2 = \infty$ for all large m if $c \geqq 3 - 6^{\frac{1}{2}}$.

In this note, we are interested in the following one-sided stopping rules, instead of the above stated two-sided stopping rules. For $c > 0$ and $1 > p \geqq 0$, define

$$s = \text{first} \quad n \geqq 1 \quad \text{such that} \quad S_n \geqq cn^p.$$

One of the results states that, if x_n are independent, $Ex_n = \mu > 0$, and $Ex_n^2 - \mu^2 = \sigma^2 < \infty$, then $Es^2 < \infty$ and

$$(1) \qquad \lim_{c\to\infty} \mu^2 Es^2/(c^2 Es^{2p}) = \lim \mu Es^2/(cEs^{1+p}) = 1.$$

When $p = 0$, $Es^2 < \infty$ implies that $P[S_1 < c, \cdots, S_n < c] = P[s > n] = o(n^{-2})$ as $n \to \infty$, which completes a result of Morimura [9]. Also (1) extends the elementary renewal theorem from first moments to second moments and generalizes some results due to Chow and Robbins [2], Hatori [6], and Heyde [7].

2. The first moment. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and x_n be a sequence of integrable random variables. Let $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots \subset \mathfrak{F}$ be Borel fields such that x_n is $\mathfrak{F}_n$-measurable and $\mathfrak{F}_0 = \{\emptyset, \Omega\}$. Put $S_n = x_1 + \cdots + x_n$, $S_0 = 0$, $m_n = E(x_n \mid \mathfrak{F}_{n-1})$ and $T_n = \sum_1^n m_j$. Assume that for some constant $\infty > \mu > 0$ and for some null set N,

$$(2) \qquad \lim_{n\to\infty} T_n/n = \mu, \quad \text{uniformly on } \Omega - N.$$

For $c > 0$ and $1 > p \geqq 0$, define

$$s = \text{first} \quad n \geqq 1 \quad \text{such that} \quad S_n \geqq cn^p.$$

Theorem 1. (i) *If for some* $0 < \delta < \mu/3$, $P[x_n \leqq m_n + n\delta] = 1$ *for all large* n, *then* $Es < \infty$.

(ii) *If* $E([(x_n - m_n)^+]^\alpha \mid \mathfrak{F}_{n-1}) \leqq K < \infty$ *for some* $\alpha > 1$ *and*

$$E(|x_n - m_n| \mid \mathfrak{F}_{n-1}) \leqq K < \infty,$$

then $ES < \infty$ *and*

$$(3) \qquad \lim_{c\to\infty} \mu Es/(cEs^p) = 1 = \lim_{c\to\infty} ES_s/(cEs^p).$$

Received 21 June 1965; revised 20 December 1965.

[1] This work has been supported by the National Science Foundation under Grant GP-04590.