A NOTE ON INVARIANT MEASURES[1]

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1. Introduction. We consider a Markov process $X_0, X_1, \cdots$ with stationary transition probability function $P(\cdot, \cdot)$ on the state space $(X, \mathbf{B})$, where X is an abstract space and $\mathbf{B}$ a countably generated Borel field of subsets of X. $P^n(\cdot, \cdot)$ denotes the nth iterate of the transition probability function and $P^0(\cdot, E)$ simply means the characteristic function of the set E.

Harris [4] introduced a recurrence condition and proved the existence of an invariant measure under such a condition. Various attempts have been made, see for instance [2], [3], and [5], to replace Harris' condition by a weaker one. The condition imposed by Isaac [5] is apparently weaker as remarked there and in [3]. The main purpose of this note is to show that Isaac's condition [5] is weaker than Harris' [4] only in a trivial sense. This is done in Section 2. This realization seems to give more insight into the results of [3] and [5]. In Section 3 we give another condition equivalent to Isaac's which seems still weaker. Some of the results of [3] are derived as consequences of these observations in Section 4.

We include some definitions and notations in this section. Most of these can be found in [1]. For any E in $\mathbf{B}$ we define

$$L(x, E) = \text{Prob}\{X_n \varepsilon E \quad \text{for some} \quad n \mid X_0 = x\},$$

$$Q(x, E) = \text{Prob}\{X_n \varepsilon E \quad \text{infinitely often} \mid X_0 = x\}.$$

The following relation can easily be verified:

$$(1.1) \qquad Q(x, E) = L(x, E) - \textstyle\sum_{n=1}^{\infty} \int_E P^n(x, dy)[1 - L(y, E)].$$

Definition 1. A nonempty set E in $\mathbf{B}$ is stochastically closed if

$$P(x, E) = 1 \quad \text{for all} \quad x \varepsilon E.$$

Definition 2. For any E in $\mathbf{B}$,

$$E^\infty = \{x : Q(x, E) = 1\}.$$

The set E^∞ is either empty or stochastically closed by Proposition 4 [1]. The following definition was introduced in [5]:

Definition 3. Let m be a σ-finite measure on $(X, \mathbf{B})$. The process is m-singular if for each x, except for an m-null set, there exists a set L_x, $m(L_x) = 0$, such that $P^n(x, L_x) = 1$ for all positive integers n. In the contrary case the process is called m-non-singular.

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