A NOTE ON UPCROSSINGS OF SEMIMARTINGALES[1]

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Let β be the number of upcrossings of an interval $[r, s]$ by an expectation-decreasing semimartingale $X_1, \cdots, X_n$.

THEOREM 1. *For each* $k = 0, 1, 2, \cdots$,

$$P(\beta > k) \leqq (s - r)^{-1} \int_{\beta=k} (X_n - r)^{-}. \tag{1}$$

PROOF. Assume without real loss in generality that $r = 0$. Let E be the event that there is a $j \leqq n$ and an $i < j$ such that $X_1, \cdots, X_i$ upcrosses $[0, s]$ at least k times and $X_j \leqq 0$, and let τ be the least such j. On the complement of E, let $\tau = n$. Plainly, τ is a stop rule. Next let $t = n$ unless, for some $j \leqq n$, $X_1, \cdots, X_j$ experiences $k + 1$ upcrossings, in which event let t be the least such j.

The inequalities in (2) below are easily checked once these three facts are verified:

(i) the event $\{\beta > k\}$ is a subevent of $F = E \cap \{X_t > s\}$;

(ii) $0 \geqq \int_E X_\tau \geqq \int_E X_t$;

(iii) the event $G = E \cap \{X_t \leqq 0\}$ is a subevent of $\{\beta = k\}$.

$$sP(\beta > k) \leqq \int_E X_t^+ \leqq \int_E X_t^- = \int_G X_t^- \leqq \int_G X_n^- \leqq \int_{\beta=k} X_n^-. \tag{2}$$

This completes the proof.

Plainly, summing over k the inequality in (1) yields Doob's result: *the expected value of* β *is bounded from above by* $\int (X_n - r)^-/(s - r)$.

Theorem 1 obviously also implies for nonnegative X_n and all k:

$$P(\beta > k) \leqq [r/(s - r)]P(\beta = k), \tag{3}$$

which is equivalent to

$$P(\beta > k) \leqq (r/s)P(\beta \geqq k). \tag{4}$$

Of course, (4) states that *for nonnegative, expectation-decreasing semimartingales, the conditional probability that* $\beta > k$ *given that* $\beta \geqq k$ *is bounded by* r/s.

Theorem 1 and its proof can be discerned in the proof of Doob's result as presented for example in [1], p. 135.

REFERENCE

[1] NEVEU, JACQUES (1965). *Mathematical Foundations of the Calculus of Probability*. Holden-Day, San Francisco.

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